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ANALYSIS AND APPLICATION OF FROZEN ORBITS FOR THE TOPEX MISSION†

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Abstract

This paper identifies and analyzes frozen orbits for use by the Topography Experiment (TOPEX) mission. Frozen orbits are characterized by almost no long-term change in eccentricity or argument of periapsis. The standard method of frozen orbit prediction is shown to be inadequate for TOPEX inclinations due to the effect of higher degree zonal harmonic gravity terms. A method is described from which long-term motion in mean eccentricity, argument of periapsis, and inclination can be predicted without numerical integration and is used to locate frozen orbits. A zonal gravity field of degree 13 is shown to be necessary and sufficient for TOPEX frozen orbit prediction. Results are verified by numerical integration methods. Frozen orbits are available for most TOPEX orbits under consideration and short-term altitude variations and maximum altitude rates are specified. Alternatives to the frozen orbit which may result in lower altitude variations and rates are briefly examined.

Nomenclature

a	mean semi-major axis
ASAP	Artificial Satellite Analysis Program
e	mean eccentricity
GM	gravitational constant times mass of Earth
g	Delaunay variable, = ω
G	Delaunay variable, = angular momentum
h	Delaunay variable, = Ω
H	Delaunay variable, = G cos(i)
H_{const}	constant value of H adopted
i	mean inclination
i_{cr}	critical inclination
i_{rep}	representative inclination
i_{var}	variable inclination
l	Delaunay variable, = **m**
L	Delaunay variable, = angular momentum of a circular orbit
LOP	Long-Term Orbit Predictor
m	mean anomaly
p	$a(1-e^2)$
R	disturbing potential
TOPEX	Topography Experiment
Ω	longitude of ascending node
ω	argument of periapsis

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I. Introduction

A frozen orbit is one whose shape and apsidal rotation remains nearly constant with time and thus provides an almost constant altitude over any particular point on a planet's surface. A frozen orbit is therefore characterized by little or no long-term change in eccentricity or argument of periapsis. The frozen orbit is attained by selection of certain values of eccentricity (**e**) and argument of periapsis (ω), for a given semi-major axis and inclination, such that the perturbing effects of the even zonal gravity harmonics on **e** and ω are balanced by those of the odd zonal gravity harmonics.

Frozen orbits can be advantageous for oceanagraphic space-based altimetry measurements since the altitude over any particular location on the sea's surface remains nearly constant lending repeatability to height measurements. Also, as the orbit decays, the basic shape of the orbit is maintained and thus there can continue to be scientific uniformity even though drag may be perturbing the orbit. The total excursions in altitude and rate of change of altitude are also nearly constant for a frozen orbit which is very desirable if these variations are within acceptable limits specified for the altimeter. A frozen orbit may require no **e** or ω correction maneuvers during the mission lifetime resulting in fuel savings and a possible simplification in operations.

However, the benefits of using the values of e and ω which result in a frozen orbit must be weighed against the alternative of using nonfrozen orbits. For example, the value of eccentricity required for a frozen orbit may result in unacceptably high altitude and altitude rate excursions in which case a nonfrozen orbit requiring periodic corrections in e might be more desirable.

In this paper, a method is described which provides a quick and inexpensive means of predicting long-term **e** and ω motion required to locate frozen orbits for most Earth orbiting spacecraft. Values of **e** and ω required to produce frozen orbits, if they exist, are identified for the range of TOPEX orbits under consideration. Short-term altitude variation and maximum rate of change of altitude are determined for the frozen orbits located and alternatives to the frozen orbit are briefly discussed.

II. Frozen Orbit Theory

The frozen orbit is attained by selection of certain values of **e** and ω, for a given semi-major axis (**a**) and inclination (**i**), such that the perturbing effects of the even zonal gravity harmonics on **e** and ω are balanced by those of the odd zonal gravity harmonics. For Earth type planets, only the perturbations due to the zonal (latitude dependent) gravity harmonics are considered significant since, with the possible exception of resonance harmonics, the spacecraft is exposed to all the sectorial and tesseral harmonics (i.e. those which have longitude dependency) equally since the Earth is spinning beneath the spacecraft. This assumption would not be valid for a slowly rotating planet such as Venus but is valid for the Earth and Mars.

For TOPEX and most other Earth orbiters, the perturbative effects of drag, solar radiation pressure, and luni-solar gravitational effects are very small relative to the Earth's gravitational perturbations and thus do not significantly affect the determination of the values of **e** and ω required to produce the frozen orbit. These and other assumptions will be verified in Section V of this paper and note that the assumptions in this paper are primarily for Earth orbiting missions with orbits similar to TOPEX.

II.1 Use of J_2-J_3 Equations to Predict Frozen Orbits

For the SEASAT mission, the values of **e** and ω required to freeze an orbit to first order for a given **a** and **i** were determined by balancing the effects of the even zonal harmonic due to J_2 with the odd zonal harmonic due to J_3 using the standard averaged equations (1) and (2).[1]

$$\frac{de}{dt} = -\frac{3}{2}\frac{J_3R^3}{p^3}(1-e^2)\ n \sin i \cos \omega \left(\frac{5}{4}\sin^2 i - 1\right) \quad (1)$$

$$\frac{d\omega}{dt} = -\frac{3}{2}\frac{J_2R^3}{p^2}\ n\left(1 - \frac{5}{4}\sin^2 i\right)$$

$$-\frac{3}{2}\frac{J_3R^3 \sin\omega}{p^3 e \sin i}\ n\left\{\left(\frac{5}{4}\sin^2 i - 1\right)\sin^2 i + e^2\left(1 - \frac{35}{4}\sin^2 i \cos^2 i\right)\right\} \quad (2)$$

Equations (1) and (2) show that the time rates of change of **e** and ω can be set to zero, resulting in a frozen orbit, by selection of a certain value of inclination or by certain combinations of **e** and ω for a given **a** and **i**. The time rate of change of **e** expressed in equation (1) can be set to zero by selection of $i = \text{asin}(\sqrt{(4/5)}) = 63.43^0$ or by selection of $\omega = 90^0$ or 270^0. The time rate of change of ω expressed in equation (2) can also be set to zero by selection of $i = 63.43^0$ since the last term in equation (2) has a neglible contribution for typical low eccentricity TOPEX orbits. The value of the inclination ($i = 63.43^0$) that causes the time rate of change of **e** and ω in equations (1) and (2), respectively, to be zero (or nearly zero) is known as the "critical" inclination (i_{cr}). Preliminary TOPEX studies[2] adopted i_{cr} as the baseline TOPEX inclination in order to attain a frozen orbit for any **e** or ω values, but a range of inclinations from 62^0 to 65^0 are currently under study due to other mission requirements.

If the inclination is not i_{cr}, which is true of most missions, the time rate of change of **e** can be set to zero by setting ω equal to 90^0 or 270^0 (see equation (1)). If the time rate of change of e is made constant by selecting $\omega = 90^0$ or 270^0, the time rate of change of ω can also be set to zero by determining the value of **e** which for a given **a** and **i** causes the time rate of change of ω expressed in equation (2) to be zero. Standard root solving methods are used and a frozen orbit is not always available at either $\omega = 90^0$ or 270^0 or at all. For example, given **a** = 7711.92 km and $i = 63^0$ (typical TOPEX values), equations (1) and (2) predict a frozen orbit if e = .00086 and $\omega = 90^0$ (no frozen orbits exist for $\omega = 270^0$).

II.2 Failure of J_2-J_3 Equations to Predict Frozen Orbits for Baseline TOPEX Orbits (Inclinations Near i_{cr})

Since the frozen orbit is produced by the balancing of the various zonal harmonic terms, the accuracy of the frozen orbit prediction depends upon the degree of the gravity field used (note that longitude dependent gravity harmonics are assumed unnecessary for frozen orbit prediction). For TOPEX orbits, the degree of the gravity field required is essentially a function of the inclination. The results of this memo and Reference 3 have found that for inclinations near i_{cr} (63.43^0) higher degree gravity harmonic terms dominate the long-term orbital motion since the perturbing effects of the low degree terms tend to offset each other near i_{cr} as equations (1) and (2) predict. Although the time rates of change of **e** and ω are very small near i_{cr}, the introduction of higher degree gravity harmonic terms can have a significant effect on the long-term **e** and ω motion.

Before further frozen orbit discussion, it is important to describe the baseline orbits under consideration for TOPEX since the accuracy of the method of standard frozen orbit prediction outlined in Section II.1 will be shown to be dependent upon the value of **i**. Table 1 lists likely nominal values of orbital elements as well as the range of values that currently comprise the TOPEX design space. For a detailed description of TOPEX mission design see Reference 4. Note that all candidate TOPEX inclinations are near i_{cr}.

Table 1 TOPEX Orbital Elements

Element	Likely nominal value	Range of values under study
a (km)	7711.92†	7678 to 7778
e	.001	0. to .001 (maximum value might increase after exact altimeter requirements are specified)
i (deg)	63.1	62 to 65
ω (deg)	90	0 to 360

†Corresponds to 10 day/127 rev repeat orbit

The effect of the higher degree gravity terms, as well as other nongravitational perturbations, upon long-term **e** and **ω** motion are typically studied by generating numerically integrated time histories of **e** and **ω**. Frozen orbits are identified by generating **e** and **ω** time histories for various initial values while keeping **a** and **i** constant. Frozen orbits are most easily identified by plotting the time histories of **e** versus **ω** creating what is referred to as an **e**-**ω** phase space.

Figure 1 illustrates frozen orbit prediction using numerical integration for a zonal field of degree 3. The plot was produced using the LOP trajectory propagation program discussed in detail in Section V.1. A closed contour in this phase space represents an oscillation of **e** and **ω** about the frozen orbit values and thus locates a frozen orbit. Further propagation using initial values of **e** and **ω** obtained from the apparent center of the closed contour, if one exists, results in another closed contour which shows a much smaller long-term variation in **e** and **ω**. The frozen **e** and **ω** values are specified to greater and greater precision in this manner until the long-term variations are as small as desired. If no closed contours are visible in the **e**-**ω** phase space, then no frozen orbits are available for the range of orbital elements investigated. Time information is obtained by plotting separate time histories for **e** and **ω**.

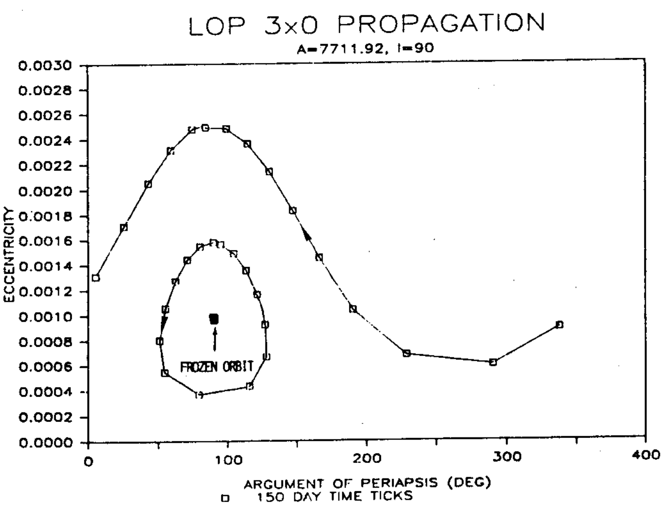

Figure 1 Sample **e**-**ω** Phase Space Obtained by Numerical Integration

Figure 2 illustrates the effects of higher degree harmonics upon the frozen orbital elements predicted using equations (1) and (2) for both TOPEX and the past SEASAT mission. Initial values of **e** and **ω** are calculated so as to freeze the orbit for a given **a** and **i** using equations (1) and (2). Note from Figure 2 that in three years time, **e** and **ω** remain nearly frozen for both SEASAT cases, but the introduction of higher degree gravity harmonics (degree 13 in the figure) cause the TOPEX **e** to rise drastically. Therefore, equations (1) and (2) are adequate for estimating the frozen orbit **e** and **ω** for SEASAT but not for TOPEX due predominantly to the proximity of TOPEX inclinations to i_{cr}.

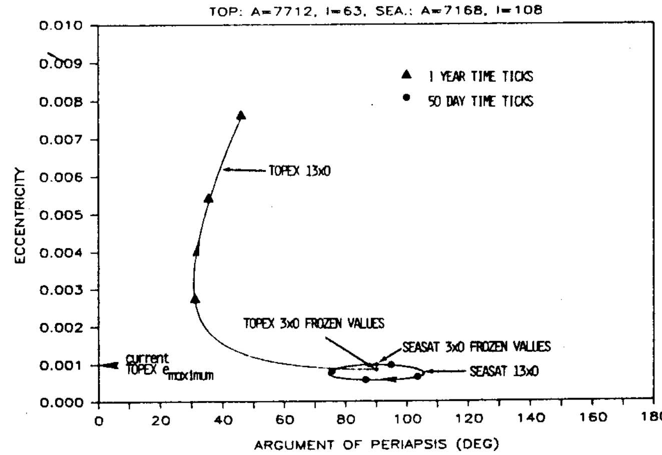

Figure 2 Effect of Higher Degree Zonal Harmonics Upon Frozen Orbit Predicted Using J_2-J_3

III. FROZEN - Mission Design Tool Developed for Frozen Orbit Prediction and Analysis

Section II demonstrated that previous frozen orbit theory, such as that utilized by the SEASAT mission, is inadequate for TOPEX studies due to the dominance of high degree zonal gravity harmonic terms which are not modelled in the simple analytical equations which only contain terms through J_3. In past TOPEX and SEASAT mission studies,[2,3] the effects of higher degree gravity harmonics, and other various perturbations, upon frozen orbit behavior were determined by performing many numerical integrations of the equations of motion creating e-ω phase spaces according to the method outlined in Section II.2.

To locate and analyze frozen orbits for the entire range of TOPEX orbits under consideration using numerical integration would be very cumbersome and expensive since the generation and processing of hundreds of long-term trajectory propagations would be required. A method of determining an e-ω phase space without numerical integration was developed by C. Uphoff[5,6,7] for studies of polar orbiting missions. Software developed by Uphoff, named SURFAS, allowed generation of an **e**-**ω** phase space, which modelled zonal gravity terms through J_5, for a constant **a** and **i** = 90°.

Two major modifications to SURFAS were required for TOPEX mission studies. Theory developed was modified by Uphoff and this author to generate **e**-**ω** phase spaces for non-polar inclinations. Subroutines from the LOP trajectory propagation program[8] were modified by this author in order that a zonal gravity field of up to degree 21 be modelled. The result is a general mission design tool which will be referred to as FROZEN which predicts long-term e, **ω**, and i motion without numerical integration and models zonal gravity terms up to degree 21.

III.1 Theory Used in SURFAS

In SURFAS[6], the canonical relationship of the derivatives of the Delaunay elements with respect to the value of the gravitational disturbing potential are employed to predict a long-term e-ω phase space without numerical integration in order to predict frozen orbits for polar orbiting missions. Zonal gravity harmonic terms through J_5 are modelled using doubly averaged analytical equations.

For reasons which will become clear, Delaunay elements[9] are used instead of classical elements to model the orbital motion and are related to the classical elements as follows:

$L = \sqrt{GM\,a}$ = angular momentum of a circular orbit; $l = m$ = mean anomaly (3a,b)

$G = L\sqrt{1-e^2} = \sqrt{GM\,a\,(1-e^2)}$ = angular momentum; $g = \omega$ = argument of periapsis (4a,b)

$H = G\cos(i) = \sqrt{GM\,a\,(1-e^2)}\cos(i)$ = Z component of angular momentum; $h = \Omega$ = longitude of the ascending node (5a,b)

GM = gravitational constant

The time rates of change of the conjugate (upper and lower case) Delaunay variables are related to each other by the derivatives of the gravitational disturbing potential (R), with respect to the conjugate variable (see equations (6) through (8)) and are the equations of motion[6,9].

$$\frac{dL}{dt} = \frac{\partial R}{\partial l}, \qquad \frac{dl}{dt} = n - \frac{\partial R}{\partial L} \qquad (6a,b)$$

$$\frac{dG}{dt} = \frac{\partial R}{\partial g}, \qquad \frac{dg}{dt} = \frac{-\partial R}{\partial G} \qquad (7a,b)$$

$$\frac{dH}{dt} = \frac{\partial R}{\partial h}, \qquad \frac{dh}{dt} = \frac{-\partial R}{\partial H} \qquad (8a,b)$$

$$n = \sqrt{GM/a^3}$$

The disturbing potential (R) is the total gravitational potential minus the point mass two-body potential. The value of **R** is a function of the orbital elements. Equations (9a,b) show that the value of R is a function of all six Delaunay and classical elements. Classical elements are also given since they are more commonly used. However, the special relationship between Delaunay elements reflected in equations (6) to (8) is required in order to generate an **e**-**ω** phase space and their use will become clear later in this section.

$$R = R(L, G, H, h, g, l) \qquad (9a)$$
$$R = R(a, e, i, \Omega, \omega, m) \qquad (9b)$$

For many mission studies, some useful simplifying assumptions pertaining to the calculation of R can be made. If the disturbing potential is singly averaged, i.e. mean anomaly (**m**) dependence removed, the value of R is no longer a function of **l**. Since the fast mean anomaly term is removed, short-term variations (i.e. over one rev) in the elements are not modelled and the value of **R** is therefore a function of average (or at least constant) orbital elements. A consequence is that semi-major axis remains constant since no nonconservative perturbations are assumed (i.e. dL/dt= 0) and thus the Delaunay element L is constant. Thus for any initial value of **L** (or **a**), the value of L remains constant and its contribution to R does not change. Equations (10a,b) reflect the functionality of **R** for a singly averaged potential.

$$R = R(G, H, h, g) \qquad L = \text{constant} \qquad (10a)$$
$$R = R(e, i, \Omega, \omega) \qquad a = \text{constant} \qquad (10b)$$

Furthermore, if additionally only zonal gravity harmonics are considered, the value of **h** (or **Ω**) no longer affects the value of **R** as shown in equations (11a,b). The potential is now doubly averaged. The value of R is not explicitly time dependent for a doubly averaged potential and is therefore constant for any initial orbit.

$$R = R(G, H, g) \qquad L = \text{constant} \qquad (11a)$$
$$R = R(e, i, \omega) \qquad a = \text{constant} \qquad (11b)$$

In order to use Delaunay variables, the relationship expressed in equations (6) to (8) must be satisfied. For the doubly averaged potential, the partial of **R** with respect to l and L is zero and thus equations (6a) and (6b) are satisfied. Since only zonal harmonics are considered, $\partial R/\partial h$ is also zero. In order for equations (8a) and (8b) to be satisfied, $\partial R/\partial H$ must also be zero. When polar orbiting missions are considered, as was the case for studies utilizing SURFAS, **H** and thus $\partial R/\partial H$ are always zero since H is a function of the cosine of **i** (equation (5a)) which is zero for $i = 90^\circ$. Equations (12a,b) summarize the effect of the additional constraint when only polar orbiters are studied.

$$R = R(G, g) \qquad L = \text{constant},\ H = 0 \qquad (12a)$$
$$R = R(e, \omega) \qquad a = \text{constant},\ i = 90^\circ \qquad (12b)$$

Thus the Delaunay variables which describe a dynamical system can be treated as having only one degree of freedom described by the variations in angular momentum per unit mass, G, and the argument of periapsis, g (in Delaunay notation). Note that for a constant **a**, **e** can be directly obtained from **G** as shown in equation 13. Another way of writing equations (7a,b) for a constant a is shown in equation (14a,b).

$$e = \sqrt{1 - G^2/(GM\,a)} \qquad (13)$$

$$\frac{-GM\,a\,e}{\sqrt{1-e^2}}\frac{de}{dt} = \frac{\partial R}{\partial \omega}, \qquad \frac{d\omega}{dt} = \frac{\sqrt{1-e^2}}{GM\,a\,e}\frac{\partial R}{\partial e} \qquad (14a,b)$$

For a given **a** and **i**, values of **e** and **ω** for which $\partial R/\partial \omega$ and $\partial R/\partial e$ equal zero can be found by plotting contours of the disturbing potential (R) in an e-ω phase space as shown in Figure 3. Note that from equations (14a,b) that if $\partial R/\partial \omega$ and $\partial R/\partial e$ are zero, then de/dt and dω/dt are zero as well which means that a frozen orbit is attained since there is no long-term change in **e** or **ω**.

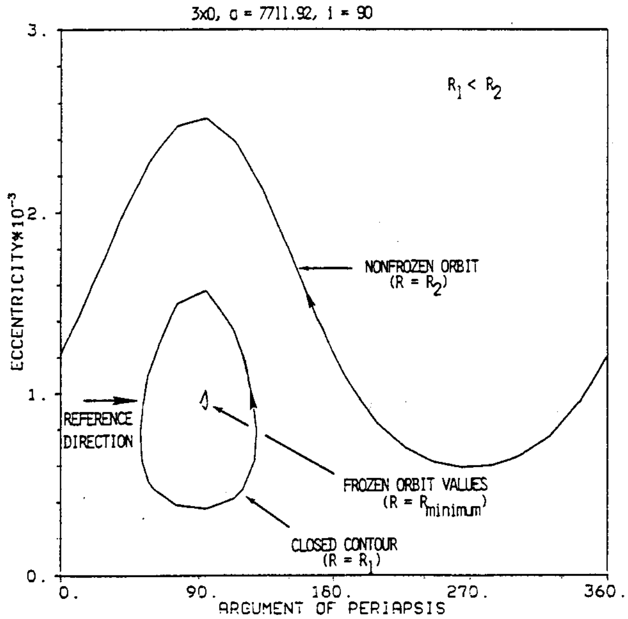

Figure 3 Sample SURFAS Output

For given values of **a** and **i**, contours of **e** and ω such as those shown in Figure 3 can be plotted. In Figure 3, only three contours of constant **R** are shown to avoid confusion but in general many contours can be plotted. If a closed contour of constant **R** appears, then the value of R reaches a maximum or minimum within the **e**-ω space. The contour plot can be viewed in an analogous manner to an altimetric relief map where closed contours containing the maximum value of R are peaks and closed contours containing the minimum value of **R** are valleys. When moving along a line of constant ω, the partial of **R** with respect to e ($\partial R/\partial e$) is obtained and $\partial R/\partial \omega$ can be obtained in a similar manner. At a peak or valley, the partials of both $\partial R/\partial e$ and $\partial R/\partial \omega$ must be zero, and by equations (14a,b), the time rates of change of e and ω must be zero resulting in a frozen orbit.

Thus the center of a closed contour in an **e**-ω space such as Figure 3 represents the values of e and ω required to freeze the orbit for given constant values of **a** and **i**. If no closed contours are present, then no frozen orbits are available for the range of orbital elements investigated. Note that in SURFAS, the value of R is dependent upon the orbital elements a, e, i, ω and the degree of the zonal gravity field modelled.

The direction of movement of **e** and ω along any contour of constant **R**, whether closed or not, in Figure 3 can be determined using equations (14a,b). The method is best illustrated by example. When moving in the direction of the boldface arrow in Figure 3, $\partial R/\partial \omega$ is negative since the value of **R** is decreasing as ω increases. From equation (14a), if $\partial R/\partial \omega$ is negative, then de/dt is positive and therefore as ω increases, **e** increases resulting in a counterclockwise circulation. The direction of motion is indicated by arrows for all the **e**-ω plots presented in this paper for convenience. Table 2 lists the inputs required to produce Figure 3.

Table 2 SURFAS Inputs Required to Produce Figure 3

Fig. 3 inputs		Explanation
7711.92	**a**	mean semi-major axis, km
90.	**i**	mean inclination, deg
3	n	degree of zonal gravity field
0.,.003	e_{min}, e_{max}	minimum and maximum mean eccentricity range
0.,360.	$\omega_{min}, \omega_{max}$	minimum and maximum mean argument of periapsis, deg

Note that since the motion is limited to one degree of freedom, **e** and ω follow the plotted contours of constant **R** since the total disturbing potential must remain constant if no nongravitational disturbing forces are introduced. From equations (14a,b), if **e** changes, ω must change correspondingly and vice-versa. Thus the long-term evolution of **e** and ω for any initial **e** and ω can be determined simply by following the contours of constant **R**! This fact yields a great deal of information about what **e** and ω will do if the frozen point is not selected or does not exist for that particular orbit and is extremely useful for mission design studies.

For example, for e_i = .0012 and ω_i = 0° in Figure 3, the orbit e-ω time histories will follow the nonfrozen orbit contour revealing that the total excursion in e will be .0019 as ω circulates around the planet. The fact that e and ω follow the contours also permits a sensitivity study as to the type of orbit which will result if targeting errors do not result in the desired mean values of **e** and ω. Unfortunately, the contours do not indicate the amount of time required for **e** and ω to move along the contours which is important for TOPEX studies since near i_{cr} the time rates of e and ω are much less that at other inclinations.

III.2 Modifications to SURFAS to Produce FROZEN

Two major modifications to SURFAS were required for TOPEX mission studies. The first modification was to add the capability to study inclinations other than 90°. The second modification was to add the capability to model up to 21 zonal gravity harmonics.

From Section III.1, the partial of R with respect to **H** ($\partial R/\partial H$) must be zero, such that **H** is constant, in order to satisfy the Delaunay relation expressed in equations (8a,b). For polar orbiters, **H** is always zero since **H** is a function of the cosine of i. However, **H** is nonzero for nonpolar orbiters and is a function of **a**, **e**, and i (equation (5a)).

When a doubly averaged potential is considered (such as that used in SURFAS and in FROZEN), the value of **a** is constant for any orbit, but the values of e and i vary with time for nonpolar orbiters. However, the value of **H** remains constant since the contribution to **H** due to variations in **e** are exactly offset by variations in i. This result can be obtained analytically by taking the derivative of **H** expressed in equation (5a) with respect to time and setting $dH/dt=0$. Solving the remaining terms for di/dt yields equation (15) which is the standard Lagrange equation for a doubly averaged potential.[10] The constancy of H was also verified by numerical integration methods.

$$\frac{di}{dt} = -\frac{de}{dt}\frac{e\cot(i)}{(1-e^2)} \qquad (15)$$

In SURFAS, at each **e** and **ω** in the e-ω space to be produced, the value of **R** was calculated using mean values of **a** and **i**. This is valid since **a** is always constant and i does not change for a polar orbiter since $di/dt=0$ for $i=90°$ as shown by equation (15). In FROZEN, if the value of **R** is calculated using constant values of **a** and **i**, contours of constant **R** will not accurately predict the long-term motion in **e** and **ω** and are therefore useless for frozen orbit prediction. This failure results since the value of **i** does not change as **e** changes in accordance with equation (15).

The following method is used in FROZEN to insure that H remains constant. A single value of H, referred to as H_{const}, is assumed for the entire e-ω space to be generated by FROZEN. Then, for each value of **e** in the e-ω space, a value of **i** is calculated such that $H=H_{const}$ using equation (16) which is equation (5) solved for **i**. The value of **i** obtained in this manner will be referred to as i_{var} (for variable i) since **i** will vary as **e** varies. The optimal value of H_{const} will be discussed later in this section.

$$i_{var} = \arccos\left(H_{const}/\sqrt{GM\,a\,(1-e^2)} \right) \qquad (16)$$

For each **e** in the e-ω space to be generated, i_{var} is used in the calculation of R such that $R=R(a, e, i_{var}, \omega)$. The e-ω space is generated as in SURFAS by connecting constant values of R. However, the output is now an e-ω-i space since i_{var} varies as e varies. The variation in **i** with **e** is shown in the FROZEN output by appending a third axis to the e-ω space which shows the value of i_{var} corresponding to each value of e. Figure 4 is a sample FROZEN output. Note that since **a** is constant, the value of inclination (i_{var}) at each e in the FROZEN output is only a function of the value of H_{const} assumed and is not dependent on the value of **ω**.

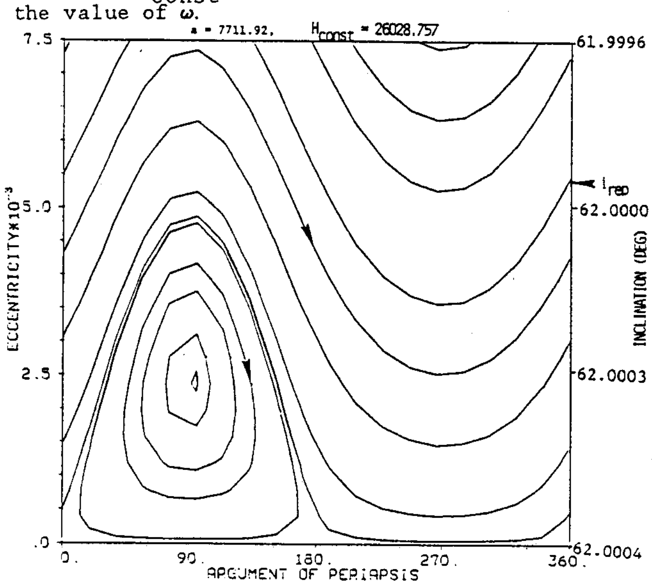

Figure 4 FROZEN Output at $i_{rep}=62°$ ($a=7711.92$ km)

The plotted contours of constant **R** in Figure 4 show the long-term motion in **e**, **ω**, and **i** for any initial combination. Note that the range of inclination plotted in Figure 4 is very small (<4.3E-4°) since for low values and small ranges of **e** investigated, the variation in i such that **H** remains constant is small. If the variation in inclination in the FROZEN output is very small, a single value of i, referred to as i_{rep}, can represent the entire e-ω-i space for convenience. A single representative value of **i** can be used since the long-term e-ω motion, such as that which would be obtained by numerical integration, will not be significantly different if the initial value of **i** is slightly different (<<.01°). Thus for the frozen orbit located in Figure 4, the value of i associated with the frozen values of **e** and **ω** can be specified as 62° rather than the exact value of 62.00033°.

To generate a FROZEN output, a single value of i_{rep} is input just as a single mean value of **i** was input in SURFAS. The optimal value of H_{const} is therefore the value which makes the maximum difference between i_{var} (equation 16) and i_{rep} in the e-ω-i space as small as possible. This is accomplished by taking the average of the values of H corresponding to the minimum and maximum **e**'s investigated using the value of i_{rep} as shown in equation (17).

$$H_{const} = \frac{H(a, e_{min}, i_{rep})+H(a, e_{max}, i_{rep})}{2} \qquad (17)$$

For the plots presented in this paper, H_{const} was calculated using equation (17). Table 3 shows the maximum difference between i_{var} and i_{rep} for various **e** ranges. Note that for small e's, which are characteristic of all TOPEX orbits, using a single representative value of **i** is a good approximation while for large values and ranges of e, the value of i_{var} corresponding to each e-ω combination in the e-ω-i space should probably be used.

Table 3 Range of i_{var} versus Range of e ($i_{rep}=63°$)

e_{min}	e_{max}	H_{const} (km^2/s)	max. $i_{var}-i_{rep}$ (deg)
0	.002	25170.774	2.91E-5
0	.015	25169.384	.00164
0	.02	25168.282	.00292
0	.1	25107.714	.0731
.09	.1	25056.640	.0140

FROZEN has the option to calculate H_{const} based on specified initial values of **e** and **i** such that the contour passing through these values will have the exact initial values desired. It may seem contradictory that the small changes in **i** as e varies are important in order for FROZEN to work but are not important when specifying TOPEX frozen orbital element sets. However, FROZEN plots contours of constant R and the value of R is very sensitive to **i**. Thus, if the **i** is not varied to make **H** constant, the contours of constant R will be totally different (and wrong) than those produced by the FROZEN method.

The second modification to SURFAS was to add the capability to model zonal gravity harmonics greater than degree 5 which was the limit in SURFAS. The doubly averaged analytical equations used in SURFAS to calculate the value of the disturbing potential given **a**, **e**, **i**, and **ω** were replaced by a group of modified subroutines from the Long-term Orbit Predictor (LOP) program.[8] The subroutines which calculate the derivatives of the potential with respect to the elements were removed from LOP and incorporated into FROZEN with a modification to calculate the value of the total disturbing potential, **R**, rather than its derivative with respect to the elements. The use of the LOP subroutines allows calculation of **R** for a zonal field of up to degree 21.

IV. FROZEN Results

Values of **e** and **ω** required to produce frozen orbits were located by generating **e**-**ω** phase spaces using the FROZEN software described in Section III. This section first summarizes the results and then presents sample **e**-**ω** phase spaces generated by FROZEN from which frozen orbit values were obtained. Table 4 lists the assumptions used to produce the **e**-**ω** phase spaces and locate frozen orbits within the TOPEX orbit design space. Verification of FROZEN results and the assumptions listed in Table 4 will be presented in Section V.

Table 4 FROZEN Assumptions for the TOPEX Mission

1) Long-term **e**-**ω**-**i** evolution follows FROZEN contours of constant disturbing potential and center of closed contour represents frozen orbit **e** and **ω** values

2) Zonal gravity field of degree 13 necessary and sufficient to model **e**-**ω**-**i** phase space in order to locate frozen orbits and for general mission design

3) FROZEN **e**-**ω**-**i** phase spaces relatively insensitive to value of semi-major axis within TOPEX design space

IV.1 Summary of Orbital Elements Required to Produce Frozen Orbit

The values of **e** and **ω** which result in frozen orbits for inclinations from 45° to 135° are summarized in Figure 5. The values of **e** and **ω** required to freeze TOPEX orbits are primarily a function of **i**. Description of the frozen orbit behavior from **i** = 45° to 90° applies to **i** = 135° to 90° due to the orbit symmetry. Frozen orbits predicted using a zonal field of degree 3 (using equations (1) and (2)) and a zonal field of degree 13 (using FROZEN) are plotted to show the sensitivity of the frozen orbit near i_{cr} (63.43°) to the degree of the gravity field.

Note from Figure 5 that for **i**'s below i_{cr}, the orbit is frozen at **ω** = 90° and the frozen **e** asymptotically rises as **i** approaches i_{cr}. A frozen **e** of .037 was located at **i** = 63.35° suggesting that frozen orbits near icr have high eccentricities. Between i_{cr} and **i** = 65.8° (the dashed line in Figure 5), the frozen **ω** is at 270° and the frozen **e** decreases from a large initial value to near zero. From **i** = 65.8° to 90°, the frozen **ω** is again at 90° and the frozen **e** grows and approaches that predicted by equations (1) and (2).

FROZEN e,ω vs INCLINATION

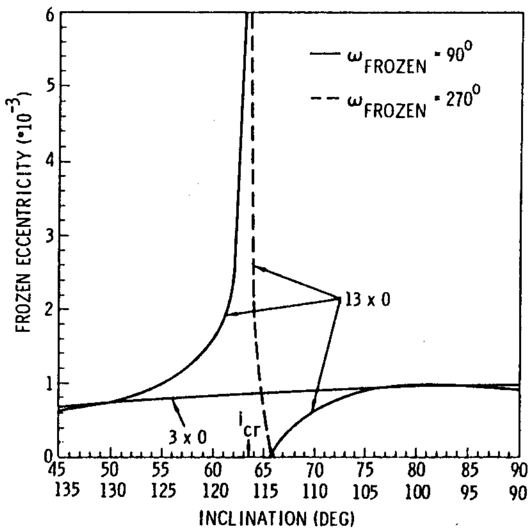

Figure 5 Frozen **e** and **ω** versus **i** (**a**=7711.92 km)

Thus for polar orbiting missions, equations (1) and (2) predict frozen **e** and **ω** values to a high degree of accuracy. For the past SEASAT mission (**i** = 108°), the frozen **e** is only slightly different when the higher degree gravity harmonics are considered as was shown in Figure 2. Figure 5 also quantitatively defines what is meant by the frequently used term "inclinations near i_{cr}". This author defines inclinations near i_{cr} as those inclinations which show a high degree of sensitivity to the size of the gravity field required to predict frozen orbit values. Though orbits with **i**'s from about 50° to 75° (105° to 130° as well) are sensitive to the size of the gravity field, **i**'s between 60° and 70° (110° to 120° as well) are drastically altered by the inclusion of a higher degree field and will therefore be referred to as "near i_{cr}".

IV.2 FROZEN e-ω Phase Spaces for TOPEX Design Space

Figures 6 to 9 are **e**-**ω** spaces generated by FROZEN for **i**'s of 63, 63.43, 64.5, and 65 degrees, respectively. The FROZEN output at **i**=62° has already been shown in Figure 4. A range of sample **i**'s from 62° to 65° is presented since as shown in Figure 5, the **FROZEN e**-**ω**-**i** space is very sensitive to inclination for the TOPEX mission. The plots were produced with the assumptions listed in Table 4 and were used to predict some of the frozen **e** and **ω** values shown in Figure 5. For a general discussion on interpretting the FROZEN output see Section III.1. Analysis and recommendations pertaining to the figures is presented in Section VI following the verification of assumptions presented in Section V.

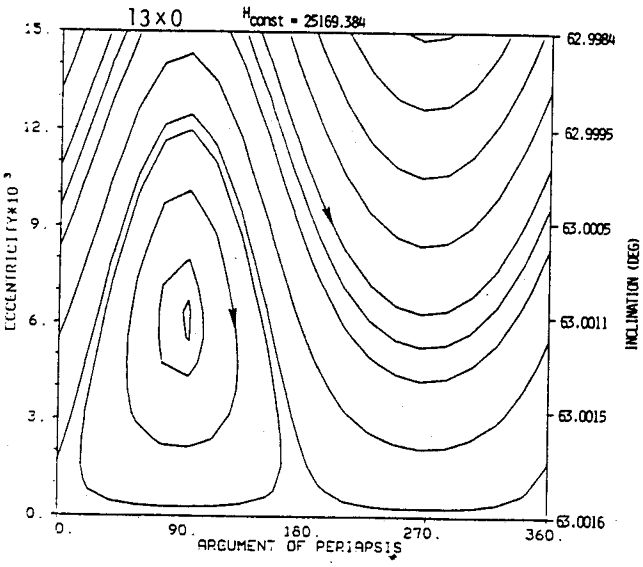

Figure 6 FROZEN Output at i_{rep}=63° (a=7711.92 km)

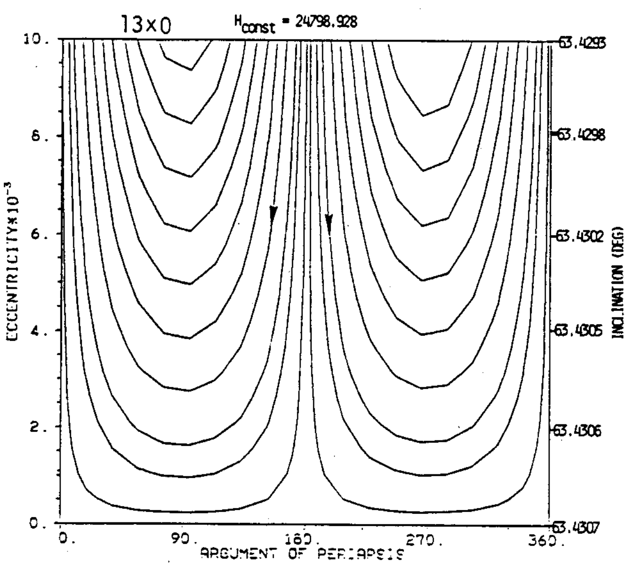

Figure 7 FROZEN Output at i_{rep}=63.43° (a=7711.92 km)

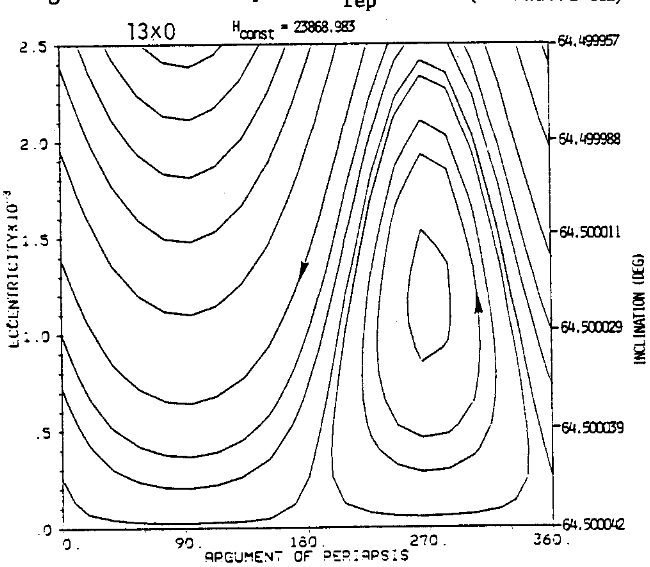

Figure 8 FROZEN Output at i_{rep}=64.5° (a=7711.92 km)

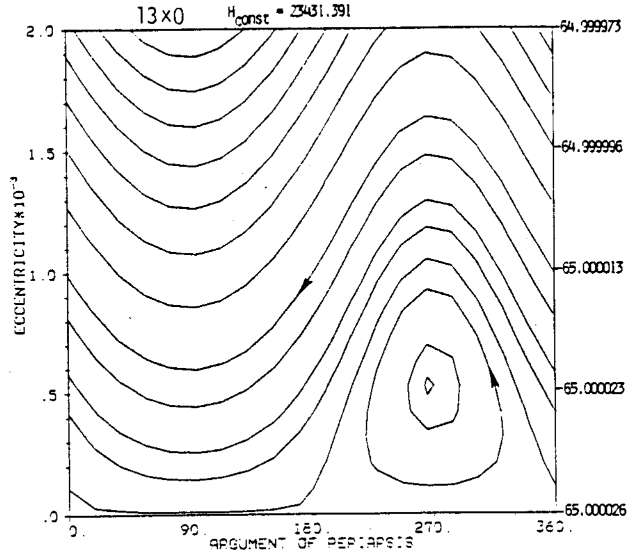

Figure 9 FROZEN Output at i_{rep}=65° (a=7711.92 km)

V. Verification of FROZEN Results and Assumptions

Verification of the assumptions listed in Table 4 which pertain to FROZEN outputs generated for the TOPEX mission are presented in this section.

V.1 FROZEN Verification by Numerical Integration

The best way to verify the **e**-ω-**i** phase space generated by FROZEN and thus assumption 1 in Table 4 is by numerically integrating the equations of motion while modelling the same degree zonal gravity harmonics. Two independent trajectory propagation programs were used for this purpose. In addition to confirming FROZEN contours, the trajectory propagation programs yielded the time required for **e** and ω to move along the contours which is the only important information not provided by FROZEN.

The Long-term Orbit Prediction (LOP) program[8] was used extensively for trajectory propagations since it can accurately predict long time spans quickly. LOP numerically integrates the Lagrange equations for average elements and can model the full 21x21 Earth gravity field, drag, solar radiation pressure, and third-body effects of the Sun and Moon. LOP ignores nonresonace tesseral and sectorial harmonics since their effect averages out in the long-term (verified in Section V.3).

A second independent trajectory propagation program, the Artificial Satellite and Analysis Program (ASAP)[11], was also used to confirm FROZEN results. ASAP propagates the osculating orbital elements and models the same perturbations as the LOP program. ASAP ,however, models the entire 21x21 gravity field and is therefore much slower than LOP. ASAP was also used to determine the short-term (over 1 rev) altitude variations and altitude rates presented in Section VI.

A sample verification of a FROZEN **e**-ω phase space by LOP for a 3x0 gravity field has already been presented in Section II, since Figure 3 generated by FROZEN is identical to Figure 1 produced by LOP. Many sample LOP verifications of FROZEN generated **e**-ω phase spaces were conducted for higher degree gravity fields and the agreement was always excellent.[1] A sample LOP verification of Figure 6 generated by FROZEN is shown in Figure 10. The initial combinations of **e** and ω, including the frozen values, were propagated for 15 years. Note that the agreement between FROZEN and LOP is excellent.

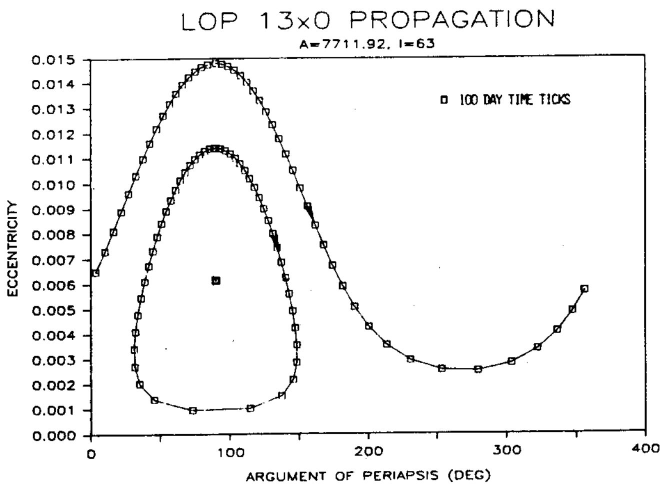

Figure 10 Sample LOP Verification of FROZEN

In 15 years, the variation in the predicted frozen value of **e** was only 8 x 10^{-6}, the variation in the frozen ω only 0.09^0, and the variation in **i** was less than $1.E\text{-}6^0$. As a result the variation in periapsis altitude was only 64 m. Thus FROZEN accurately located the frozen orbit values. Note that the time rate of change of e and ω in Figure 10 is very small and that the time required for **e** and ω to move along the large closed contour is about 15 years. Also note that **i** is frozen since **e** is frozen from equation (15). The significance of these low rates of change will be discussed in Section VI.

A sample verification of a portion of the FROZEN output shown in Figure 6 by the ASAP trajectory propagation program is shown in Figure 11. Since ASAP propagates osculating orbital elements, the initial mean values of **a**, **e**, **i**, and ω obtained from Figure 6 must be converted to osculating elements using an ASAP utility routine. The conversion takes into account variations in the elements due to J_2.

Figure 11 plots the osculating values obtained at the ascending node in order to mask the short-term variations which would obscure long-term **e** and ω trends. Note that mean values of e and ω are calculated from the osculating values at selected points to show that the ASAP output is in excellent agreement with the FROZEN result.

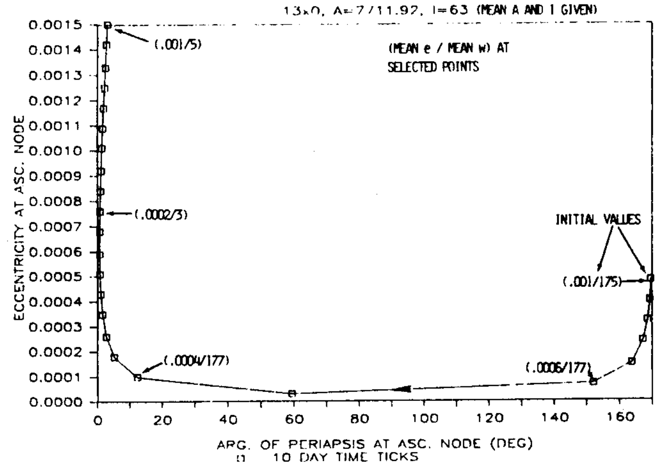

Figure 11 Sample ASAP Verification of FROZEN

V.2 FROZEN Sensitivity to Value of a

Assumption 3 in Table 4 that the **e**-ω-**i** phase spaces generated by FROZEN are relatively insensitive to the value of **a** will now be verified. Table 5 illustrates the relative insensitivity of the frozen orbit **e** and ω values to the value of **a**.

Table 5 Sample Sensitivity of Frozen e and ω values to Value of Semi-major Axis

Frozen e/ω values as a function of a

i (deg)	a minimum = 7678 km	a nominal =7711.92 km	a maximum = 7778 km
62	.00246/90	.00242/90	.00236/90
65	5.4E-4/270	5.2E-4/270	4.8E-4/270

Frozen orbit values for the minimum, nominal, and maximum values of **a** were calculated using FROZEN. Thus the results obtained using the nominal value of **a** = 7711.92 km will not be very different than those obtained using other values of **a** within the design space, but FROZEN should be run with the exact initial value of **a** when a final value is selected.

V.3 Force Modelling Required for FROZEN Orbit Prediction

This section verifies assumption 2 in Table 4 that a zonal gravity field of degree 13 is necessary and sufficient to predict frozen orbits for TOPEX. The size of the zonal field required is first determined using FROZEN and for the frozen values located, the perturbing effects of sectorial and tesseral Earth gravity harmonics, drag, solar radiation pressure, and third body effects of the Sun and Moon upon the frozen orbit are invesitgated by numerical integration methods.

The maximum zonal field required to model TOPEX orbits can be determined by running FROZEN for the same orbital state and increasing the degree of the gravity field until contours of constant disturbing potential in the FROZEN output stop changing significantly. As an example of the sensitivity of the frozen orbit to the degree of the gravity field, FROZEN was run at $i = 65^0$ for a zonal field from degree 2 to 21 and the results are summarized in Figure 12.

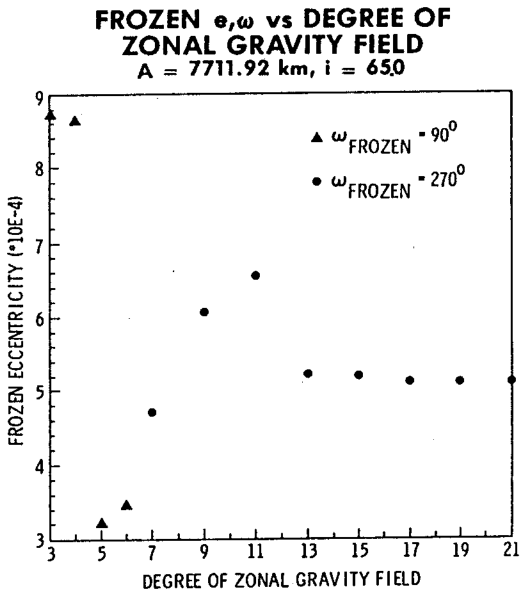

Figure 12 Frozen e and ω versus Degree of Zonal Gravity Field

Note that when only J_2 is considered, no frozen orbit is available since the frozen orbit is produced by the balancing of the even and odd zonal harmonics. The frozen values of e and ω for a zonal field of degree 3 agree with those predicted by equations (1) and (2). Inclusion of zonals through degree 5 lowers the frozen e value. Inclusion of zonals through degree 7 shifts the frozen ω from 90° to 270°. Zonals through degree 13 lower the frozen e but as Figure 12 shows, inclusion of zonals higher than degree 13 has little effect. Note that the inclusion of even zonals beyond degree 2 has little effect upon the frozen orbit values emphasizing that the principle disturbance comes from the equatorial asymmetry of the Earth's mass distribution which is represented by the odd zonal harmonics.

The effects of the other gravitational and nongravitational perturbations upon the frozen orbit must be investigated in order to determine if the frozen orbit is realistically achievable, i.e. that the zonal gravity harmonics are the dominant perturbing force. To study these effects, frozen orbits determined using FROZEN with a zonal field of degree 13 were propagated using the LOP trajectory propagation program discussed in Section V.1 for various force modelling. As an example, the TOPEX frozen orbit predicted by FROZEN at $i = 63^0$ was propagated for 3 years with various force modelling and the results are summarized in Table 6.

Table 6 Sample Effects of Various Perturbations Upon Frozen Orbit

Case #	LOP perturbations modelled	Variation (var.) in element over 3 year period = maximum value - minimum value a var. (m)	e var. *10.E-6	ω var. (deg)	i var. (deg) *10.E-3	altitude at periapsis (m)
1	13x0 gravity	0.	8.	.09	.0	64.
2	21x0 gravity	0.	8.	.09	.0	64.
3	13x13 gravity	89.	13.	.20	4.6	166.
4	13x13 gravity*	15.	<.1	.005	.0	16.
5	21x21 gravity	5.	93.	.88	0.3	723.
6	drag	227.	2.	.00	.0	209.
7	SRP**	8.	147.	1.17	.4	1144.
8	Sun as 3rd body	0.	3.	.09	5.9	30.
9	Moon as 3rd body	0.	15.	.70	15.3	120.
10	21x21 gravity, drag, SRP, Sun, Moon	224.	89.	1.43	6.1	892.

* Generated by ASAP, variations at ascending node, length of propagation = 1 year
** SRP = solar radiation pressure

Table 6 lists the variations encountered over 3 years in specified mean elements. Note that the variations in e and ω when only the zonal field of degree 13 is modelled (case 1) are extremely small. Inclusion of zonals to degree 21 (case 2) has little effect upon the frozen orbit Note from Section V.1 that the LOP program ignores nonresonance sectorial and tesseral harmonics since their effect averages out in the long-term. This assumption was verified using the ASAP trajectory propagation program as shown in case 4. The variation in the values at the ascending node are listed since variations at the ascending node reflect variations at any other point in the orbit. A shorter time span is propagated in case 4 since ASAP is much slower than LOP in run time. The inclusion of all the 13th order resonance terms (case 5) has a noticable but small effect upon the orbit.

Note from Table 6, that solar radiation pressure has the largest effect on the frozen orbit values but that the orbit is still very frozen. Drag causes a to decrease about 75 m per year but has little effect upon e or ω. Case 10 models all major perturbations and Figure 13 plots e versus ω for this case. In three years, e ranged from .00612 to .00621 and ω ranged from 88.77° to 90.21°. Thus the orbit is nearly frozen and variation in periapsis altitude was less than 1 km over the three years. It might be possible to improve the frozen orbit values obtained by FROZEN by propagation with full force modelling but the improvement would be small and possibly difficult since the orbital elements do not change in a smooth fashion when all major perturbations are modelled.

VI. Analysis of TOPEX Frozen Orbits

As stated in the introduction, the guaranteed benefit of a frozen orbit for TOPEX is a nearly constant altitude over any point on the ocean's surface and the virtual elimination of e and ω correction maneuvers. Two additional parameters which directly affect altimeter performance are the variation in altitude (maximum altitude-minimum altitude) and the maximum rate of change of altitude, both evaluated over one orbit. These two parameters are also nearly constant for a frozen orbit but the magnatudes are a function of

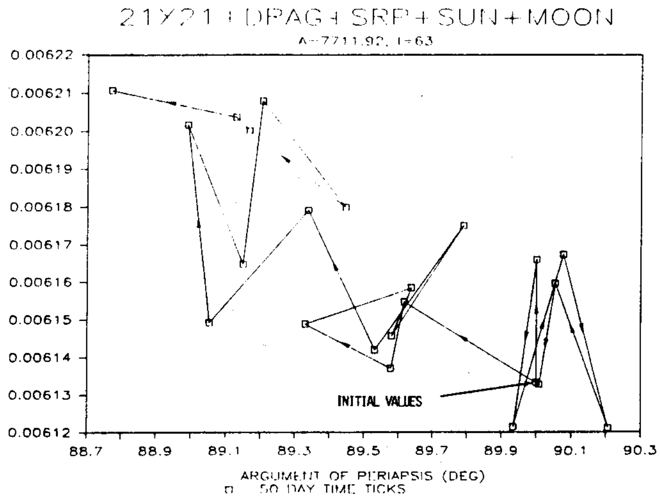

Figure 13 Effect of Other Perturbations upon FROZEN orbit values

the orbital elements required to produce the frozen orbit. Thus, the frozen **e** may result in altitude variations and altitude rates which exceed the allowable tolerances specified for the altimeter. Therefore, the short-term (over one rev) variations in the orbit must be related to the mean frozen orbit elements in order to evaluate whether the frozen orbit is desirable for the TOPEX mission.

Figure 14 summarizes the frozen orbit values for the TOPEX design space and relates the altitude variation and maximum altitude rate to selected frozen values. Remember that these values will not be significantly affected by the value of **a** within the TOPEX design space. The altitude variation and maximum altitude rates were obtained by finite differencing of the output of the ASAP trajectory propagation program described in Section V.1 while modelling J_2. Selected sets of mean frozen orbital elements were converted to osculating elements for input to ASAP by the method outlined in Section V.1. The altitude variation and rate are the same for $\omega = 90°$ or $270°$.

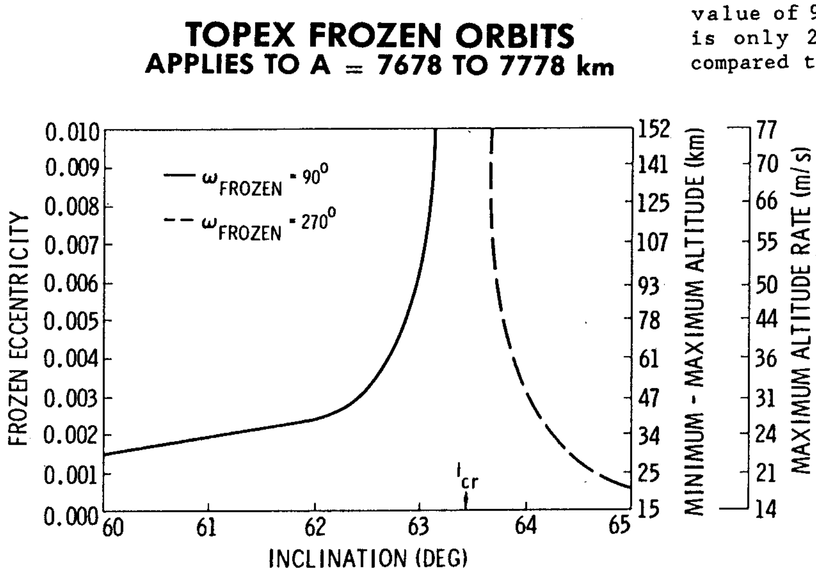

Figure 14 Frozen Orbital Elements for TOPEX Design Space

Note from Figure 14, that frozen eccentricities below the currently specified maximum allowable **e** (e_{max}) of .001 are only available for **i**'s from 64.6° to 65°. Frozen orbits for **e**'s below .002 are only available for i's from 64.2° to 65°. However, the current value of e_{max} is somewhat conservative since exact altimeter requirements are not yet specified and thus a larger e_{max} might be acceptable in the future. However, it does appear that frozen orbits may not be desirable for the entire design space due to the large values of **e** required to freeze the orbit at certain **i**'s.

However, alternatives to the frozen orbit exist due to the low time rates of change of **e** and ω within the TOPEX design space and can be investigated with the aid of the FROZEN software. Since a detailed discussion of alternatives to the frozen orbit is beyond the scope of this paper, one example of a possible alternative to the frozen orbit will be given to illustrate that the initial values of **e** and ω are very important for nonfrozen orbits to minimize altitude variation and altitude rate.

For a given e_{max}, values of e and ω can be selected using the FROZEN program such that **e** initially decreases to near zero and then increases until e_{max} is reached where an **e** correction maneuver must be performed. Numerical integrations have shown that the time before an **e** correction is required can be hundreds of days and is a function of the specified e_{max} and **i**. This type of orbit will be referred to as a decreasing **e** orbit and an example of such an orbit is shown in Figure 15.

Figure 15 plots a decreasing **e** orbit for $i = 63°$ and the value of e_{max} is assumed to be 0.001. Using the FROZEN output for $i = 63$ (Figure 6), initial values of $e = .001$ and $\omega = 172°$ are obtained such that the **e** initially decreases to near zero before increasing to e_{max} at which time an **e** correction would be required. These values are then propagated using the LOP trajectory propagation program (Section V.1). Values of altitude variation and maximum altitude rate are obtained at selected points by the method used in Figure 14. Note that the maximum altitude variation is only 24 km as compared to the frozen value of 94 km. Also, the maximum altitude rate is only 20 m/s for the decreasing **e** orbit as compared to 50 m/s for the frozen orbit.

The **e** for the orbit propagated in Figure 15 remains below .001 for almost 250 days at which time an **e** correction maneuver would be required. If the value of e_{max} were greater than .001, this time would be longer. It is useful to note that the predicted maneuver frequency for **a** corrections due to drag effects ranges from about 65 to 425 days depending primarily on the solar flux and the area to mass ratio of the spacecraft.[2] Thus drag correction maneuvers could be required more frequently than **e** correction maneuvers for the decreasing **e** orbit at which time **e** could be restored to a nominal value. Also Reference 1 shows that the time rate of change of **e** decreases as **i** increases from 62° to 65° causing the time before **e** correction maneuvers are required to be even greater for the higher **i**'s in the design space.

VII. Conclusions

Standard frozen orbit theory which models gravity potential terms through J_3 has been proven to be inadequate for TOPEX mission planning. For typical TOPEX orbits, a zonal field of degree 13 was required to accurately predict the orbital motion and locate frozen orbit values due to the proximity of TOPEX inclinations to the critical inclination. A versatile mission design tool, FROZEN, was described and was used to locate frozen orbits. FROZEN quickly and inexpensively predicts long-term motion in **e**, **ω**, and **i** without numerical integration and was verified by two independent trajectory propagation programs. Perturbations due to tesseral and sectorial gravity harmonics, drag, solar radiation pressure, and third-body effects of the Sun and Moon were shown to have only minor effects on the frozen orbit values.

Frozen orbits are available for most TOPEX orbits under consideration but the value of eccentricity required to freeze the orbit is higher than the currently specified maximum value of .001 for the majority of the design space. A possible alternative to the frozen orbit, the decreasing **e** orbit, was discussed and offers reduced altitude vairation and lower altitude rates for most TOPEX orbits but requires periodic **e** corrections.

Whether the frozen orbit is desirable for the TOPEX mission is dependent upon final altimeter requirements which have not yet been finalized. Two important altimeter performance parameters, short-term altitude variation and rate of change of altitude, were specified for the frozen orbits located to aid in future evaluation. Regardless of the final TOPEX orbit selected, this paper has shown that the initial values of **e** and **ω** selected have a significant effect upon the long-term orbital motion.

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