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AN ACCURATE AND EFFICIENT SATELLITE LONG–TERM ORBIT PREDICTOR EMPLOYING "FICTITIOUS" MEAN ORBITAL ELEMENTS*

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Abstract

For purposes of any precise trajectory predictions involving perturbing forces, it is necessary to use differential formulations either in the Cartesian coordinates as the integration parameters (e.g., in the Cowell's method) or in the classical osculating orbital elements as the integration parameters (e.g., in the Lagrangian equations). In general, any trajectory processes that are required to reveal instantaneous or short–term variations must use the rectangular coordinates, since the osculating elements of Lagrange's Equations are simply a convenient perturbation transformation derived from the original differential equations of motion in the rectangular coordinates. However, the Lagrangian equation set does contain mathematical singularities at zero eccentricity and zero inclination. For satellite orbits, the rectangular coordinates and some of the osculating elements are considered as fast variables and therefore require very small integration steps in order to account for short–term or instantaneous variations of the integration parameters at the expense of very long integration time. The harmonics analysis of the gravitational perturbation potential used in the satellite equations of motion is originally obtained in rectangular coordinates and therefore is well–suited for applications either in the rectangular coordinates of Cowell's method or in the osculating elements of Lagrange's Equations after tedious series transformations of the perturbing potential in terms of the osculating elements. For preliminary satellite mission designs, however, it is not cost–effective to use such costly time–consuming integration programs.

By using Von Zeipel's generating function procedure to average the perturbing Earth gravitational potential, with respect to the fast variable (mean anomaly), a set of "fictitious" mean orbital elements is obtained that is shown to be a function of the nonlinear J_{20}^2 (square of the 2nd zonal harmonic coefficient) term. On the assumption of appropriate conversion from a set of "fictitious" initial mean elements to an equivalent set of initial osculating elements or vice versa, we have demonstrated that the long–term orbit prediction using the "fictitious" mean elements is as accurate as that using the osculating elements, but has a computing speed about two orders of magnitude faster.

If the J_{20}^2 terms were neglected in the mean orbital elements approach, the accuracy of the approach is decreased by about two orders of magnitude due to the effect of missing J_{20}^2 terms.

For short–term orbit predictions, the osculating elements approach must be used.

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I. Introduction

For any preliminary mission designs for either interplanetary or Earth satellite missions, software tools that are not only efficient and cost–effective but also adequately accurate are preferred rather than those high–precision software programs that numerically integrate trajectories at the expense of extremely long integration time, as well as high cost.

In interplanetary cases, preliminary mission designs usually have been carried out by employing simple conic, patch–conic, or multi–conic approximate methods. Recently, Jet Propulsion Laboratory (JPL) developed the single–step and multi–step Plato[1] programs that combine the multi–conic technique with the minimization of total impulsive ΔV for multiple–flyby trajectories, with constraints on flyby parameters and maneuver times. The Plato programs, in general, give adequately accurate preliminary design trajectories. A final high–precision numerical integration program is always used either (a) to verified the accuracy of the preliminary design, or (b) to fine–tune the preliminary design to the prescribed accuracy.

For Earth satellite missions, most preliminary mission designers have been using high–precision numerical integration programs (in osculating orbital elements or Cartesian coordinates) as their design tool, at the expense of high cost and very long preliminary design time spans. In fact, some preliminary satellite mission designs (e.g., frozen orbit designs) can not be accomplished satisfactorily in high–precision programs in osculating orbital elements (to be explained later). Recently, JPL has developed the SAMDP program[2] in fictitious mean orbital elements, and the LOP program[3] (same as SAMDP but more efficient due to coding streamlining). LOP is extremely well–suited for preliminary satellite mission designs in the sense of cost–effectiveness, and especially so for frozen orbit designs at or near the critical inclination. When done properly, programs in mean elements eliminate the nodal–to–nodal short–term periodic variations without affecting the long–term orbit prediction accuracy. Because of the elimination of short–term variations, LOP can take long integration steps (in tens instead of hundredths of revolution) without introducing integration errors. However, comparisons of results of long–term predictions from the LOP program with corresponding results from the high–precision ASAP[4] or DPTRAJ[5] programs show some significant deviations. It is the purpose of this paper to trace the cause of deviations and to reformulate the LOP in such a way that the deviations are reduced to an acceptable, prescribed, vanishing minimum.

The method of formulation in the LOP program in mean orbital elements is basically that of Kaula[6], together with a transformation to eliminate the "mathematical singularity" at zero eccentricity in the Lagrangian equations. In an averaging process for perturbing gravitational harmonics via mean anomaly in order to express the Lagrangian equations in terms of mean elements instead of the original osculating elements, Kaula and thus the LOP program failed to include the dominant non–linear harmonic effect (J_{20}^2 for Earth) into the formulation.

The following section will attempt to explain the necessity of J_{20}^2 terms in the mean elements Lagrangian equations and to show briefly how to obtain the J_{20}^2 terms in the Lagrangian equations for purposes of integration.

II. Theoretical Formulations

The Lagrange's planetary equations in osculating classical orbital elements are:

$$\left.\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{\partial R^*}{\partial M},\\
\frac{de}{dt} &= \frac{1-e^2}{na^2e}\frac{\partial R^*}{\partial M} - \frac{(1-e^2)^{\frac{1}{2}}}{na^2e}\frac{\partial R^*}{\partial \omega},\\
\frac{de}{dt} &= -\frac{\cos\ i}{na^2(1-e^2)^{\frac{1}{2}}\ \sin\ i}\frac{\partial R^*}{\partial i} + \frac{(1-e^2)^{\frac{1}{2}}}{na^2e}\frac{\partial R^*}{\partial e},\\
\frac{di}{dt} &= -\frac{\cos\ i}{na^2(1-e^2)^{\frac{1}{2}}\ \sin\ i}\frac{\partial R^*}{\partial \omega} - \frac{1}{na^2e(1-e^2)^{\frac{1}{2}}\sin\ i}\frac{\partial R^*}{\partial \Omega},\\
\frac{d\Omega}{dt} &= \frac{1}{na^2(1-e^2)^{\frac{1}{2}}\sin\ i}\frac{\partial R^*}{\partial i},\\
\frac{dM}{dt} &= n - \frac{1-e^2}{na^2e}\frac{\partial R^*}{\partial e} - \frac{2}{na}\frac{\partial R^*}{\partial a}.
\end{aligned}\right\} \quad (1)$$

where R^* is the instantaneous disturbing potential of an aspherical planet and is expressed usually in terms of spherical coordinates, as shown in Equation (2) below. Kaula[6] has transformed it in terms of the classical osculating element coordinates a, e, i, ω, Ω, M (or f), as shown in Equation (3) below.

$$R^* = \sum_{\ell=2}^{\infty}\sum_{m=o}^{\ell} V_{\ell m} = \sum_{\ell=2}^{\infty}\sum_{m=o}^{\ell} \frac{\mu R_e^{\ell}}{r^{\ell+1}} P_{\ell m}(\sin\Phi)$$

$$[(C_{\ell m}\ \mathrm{Cos}\,m\lambda) + S_{\ell m}\mathrm{Sin}\,m\lambda)] \quad (2)$$

$$V_{\ell m} = \frac{\mu R_e^{\ell}}{a^{\ell+1}} \sum_{p=o}^{\ell} F_{\ell mp}(i) \sum_{q=-\infty}^{\infty} G_{\ell pq}(e)\, S_{\ell mpq}$$

$$(\omega, M, \Omega, \Theta) \quad (3)$$

where

$$S_{\ell mpq} = \begin{bmatrix} C_{\ell m} \\ -S_{\ell m} \end{bmatrix}_{\ell-m\ \mathrm{odd}}^{\ell-m\ \mathrm{even}} \mathrm{Cos}[(\ell-2p)\omega + (\ell-2p+q)$$

$$M + m(\Omega-\Theta)]$$

$$+ \begin{bmatrix} C_{\ell m} \\ S_{\ell m} \end{bmatrix}_{\ell-m\ \mathrm{odd}}^{\ell-m\ \mathrm{even}} \mathrm{Sin}[(\ell-2p)\omega + (\ell-2p+q)$$

$$M + m\,(\Omega-\Theta)] \quad (4)$$

R_e is the equatorial radius of the Earth or any planet,

and Θ is the hour angle (Greenwich Sidereal Time) of the satellite in question.

For purposes of any precise trajectory predictions involving perturbing forces, it is necessary to use differential formulations either in the rectangular coordinates as the integration parameters (e.g., in the Cowell's method) or in the classical osculating orbital elements as the integration parameters, as shown in the Lagrangian equations in Equation (1). In general, any trajectory processes that are required to reveal instantaneous or short–term variations usually must use the rectangular coordinates, since the osculating elements of Lagrange's Equations are simply a convenient perturbation transformation derived from the original differential equations of motion in the rectangular coordinates. For satellite orbits the rectangular coordinates and some of the osculating elements are considered as fast variables and therefore require small integration steps in order to account for short–term or instantaneous variations of the integration parameters. In Equation (2), the harmonics analysis of the gravitational perturbation potential R^* used in the satellite equations of motion is mathematically exact in the instantaneous sense according to the general potential theory. It is originally obtained in rectangular coordinates and therefore is well–suited for applications either in rectangular coordinates of Cowell's method or in the osculating elements of Lagrange's Equations after tedious series transformations of the disturbing potential in terms of the osculating elements. The integration of the equations of motion in either rectangular coordinates or osculating elements demands small integration steps to obtain instantaneous or short–term precisions at the expense of excessively long integration time for general applications. For preliminary satellite mission designs, it is not cost–effective to use such time–consuming integration programs as ASAP or DPTRAJ. The method of using a set of "fictitious" mean orbital elements for orbit long–term predictions is a way to obtain cost–effectiveness at the expense of missing the short–periodic accurate information.

In order to remove the fast variable, the mean anomaly M, and thus obtain the "average" or "mean" long–periodic and secular terms of a satellite trajectory, the instantaneous disturbing potential R^* is "averaged" with respect to the short–periodic variable M from 0 to 2π. It can be shown that this averaging process is mathematically equivalent to setting the coefficient of M in Equation (4) equal to zero; i.e.

$$\ell - 2p + q = 0 \quad \text{or} \quad q = 2p - \ell \quad (5)$$

Thus the averaged R^* is independent of M now. Accordingly, the orbital elements a e, i, ω, Ω, and M in Equations (1) are no longer osculating elements, but become "fictitious" mean elements when R^* is an averaged perturbing potential. From this point on in this paper, we shall assume R^* is the averaged one!

The Lagrangian equations (1) contain mathematical singularities both at e = 0 and i = 0. For most Earth–observing satellites, inclinations are usually high enough to have global coverage of the planetary surface and thus singularities at inclinations at or near zero are rarely real problems. Circular or near–circular orbits are usually preferred for planetary satellites. To eliminate the singularity at zero eccentricity in Lagrange's Equations, the following transformations of variables from e and ω to h and k are introduced:

$$\begin{aligned} h &= e \sin\omega \\ k &= e \cos\omega \end{aligned} \tag{6}$$

A new variable, λ_N, called "stroboscopic mean node" is also introduced to replace the variable M. Equations (1) then are transformed to the following new set of planetary equations in terms of the mean elements a, h, i, k, Ω, and λ_N[3].

$$\left.\begin{aligned}
\frac{da}{dt} &= \frac{2}{na}\frac{1}{S_o}\frac{\partial R}{\partial \lambda_N}, \\
\frac{dh}{dt} &= \frac{(1-e^2)^{\frac{1}{2}}}{na^2}\frac{\partial R}{\partial k} - \frac{k \cot i}{na^2(1-e^2)^{\frac{1}{2}}}\frac{\partial R}{\partial i} - \frac{h\ (1-e^2)^{\frac{1}{2}}}{na^2 S_o}\beta'\frac{\partial R}{\partial \lambda_N}, \\
\frac{di}{dt} &= \frac{\cot i}{na^2(1-e^2)^{\frac{1}{2}}}\left[k\frac{\partial R}{\partial h} - h\frac{\partial R}{\partial h} + \frac{1}{S_o}\frac{\partial R}{\partial \lambda_N}\right] \\
&\quad - \frac{1}{na^2(1-e^2)^{\frac{1}{2}}\sin i}\left[+\frac{\partial R}{\partial \Omega} + \frac{\partial R}{\partial \lambda_N}\right] \\
\frac{dk}{dt} &= -\frac{(1-e^2)^{\frac{1}{2}}}{na^2}\frac{\partial R}{\partial h} + \frac{h \cot i}{na^2(1-e^2)^{\frac{1}{2}}}\frac{\partial R}{\partial i}\frac{k\ (1-e^2)^{\frac{1}{2}}}{naS_o}\beta'\frac{\partial R}{\partial \lambda_N}, \\
\frac{d\Omega}{dt} &= \frac{1}{na^2(1-e^2)^{\frac{1}{2}}\sin i}\frac{\partial R}{\partial i}, \\
\frac{d\lambda_N}{dt} &= \frac{n}{S_o} - \frac{d\Theta}{dt} + \frac{1}{na^2 S_o}(1-e^2) + \beta'\left[h\frac{\partial R}{\partial h} + k\frac{\partial R}{\partial k}\right] \\
&\quad - 2a\frac{\partial R}{\partial a} + \frac{S_o - \cos i}{(1-e^2)^{\frac{1}{2}}\sin i},
\end{aligned}\right\} \tag{7}$$

where

$$\lambda_N = (M + \omega)/S_o + (\Omega - \Theta)$$

$$n = \left[\frac{\mu}{a^3}\right]^{\frac{1}{2}}$$

$$e = (h^2 + k^2)^{\frac{1}{2}}$$

$$\beta' = \frac{1}{1 + (1-e2)^{\frac{1}{2}}}.$$

$$S_o = P/Q$$

and P and Q are relative prime integers and P/Q approximates the number of orbits per planet revolution.

Note R (a, h, i, k, Ω, λ_N) is the transformed disturbing potential function of the averaged R*(r, ϕ, λ or a, e, i, Ω, ω, M) of Equations (2), (3), and (4) together with the constraint $\ell - 2p + q = 0$ of Equation (5) such that the short–term periods have been removed. The solutions of Equations (7) are supposed to yield correct "mean" orbital elements containing only the secular and long–periodic terms. However, because of the reasons to be explained in what follows, solutions of Equations (7) are not as accurate as they should be due to the missing nonlinear effect J_{20}^2 terms in the averaged disturbing potential function R* or R.

For integration with respect to mean orbital elements, the conventional harmonic potential expansion (such as in Equations (2) or (3)) alone is no longer appropriate or adequate, because the expansion is in a form to be used exclusively for osculating elements. When mean elements are used as the variables for integrations, correct or accurate approximate solutions can be expected, in general, in two lines of thoughts. Firstly, we can express the system Equations (1) as

$$\frac{dx}{dt} = \epsilon f(t,x), \qquad x(t_o) = x_o(e_o,\omega_o,i_o,etc) \tag{8}$$

where the small parameter ϵ is of the order of the dominant J_{20} in the case of Earth. A power series solution in ϵ can be obtained for the system Equation (8) via successive iterations, and thus the ϵ^2 term will appear as the largest correction term. This method, however, is too complex and impractical to be carried out mathematically. Secondly, we can expand the conventional disturbing potential expression for osculating elements in a Taylor series expansion about the mean elements, thus the J_{20}^2 term will also appear as the largest correction term in the series expansion of the potential. We have not tried this Taylor expansion method here due to its complexity, but note that Brouwer's[7] transformation of the original Hamiltonian (use only zonals) by Von Ziepel's method of generating function in Delaunay variables is, in fact, an averaging method, and thus is very akin to this expansion method about the mean elements. Accordingly, we will make use of the same Von Ziepel's method of generating function to obtain the average disturbing potential that includes the sought– after nonlinear J_{20}^2 terms in the case of Earth. For other planets the dominant perturbing harmonic might not be J_{20} but J_{40}, for example, and thus the sought–after nonlinear terms would be those with J_{40}^2 instead of J_{20}^2. For Earth's case the magnitude of J_{20}^2 is comparable to J_{40} and thus can not be ignored for Earth satellite missions. This is especially so for the TOPEX/POSEIDON mission, due to its extremely stringent measurement accuracy requirements. The approximate analytic average (or mean) elements solution of Brouwer has two limitations: (A) confined to applications for zonal harmonic up to J_{50} only, and (B) singular solutions at or near the critical inclination due to his inappropriate approximations $\dot{\omega} = \dot{\omega}_s$ (instead of the exact relation $\dot{\omega} = \dot{\omega}_s + \dot{\omega}_\ell$), where $\dot{\omega}$, $\dot{\omega}_s$, and $\dot{\omega}_\ell$ are, respectively, the rate of change of argument–of–periapsis, secular rate of change of argument–of–periapsis and long–periodic rate of change of argument of periapsis. Because of the approximations, these analytic solutions of Brouwer have not been widely used for long–term orbit predictions.

The Von Zeipel procedure as a method of averaging yields a set of differential equations in mean elements a, e, i, Ω, ω, and M via an average disturbing potential exactly as those shown by Kaula[2] except the J_{20}^2 terms that are present only in the Von Ziepel's method of generating function. Without repeating main portions of the differential equation set that represent the well–known conventional linear perturbations due to harmonics of all degrees and orders, we show below only the sought–after nonlinear J_{20}^2 portions of the differential equation set:

$$\left.\begin{aligned}
\frac{de'}{dt} &= A \sin 2\omega \\
\frac{d\omega'}{dt} &= B + C \cos 2\omega \\
\frac{di'}{dt} &= -\frac{e \cot i}{1-e^2}\left[\frac{de'}{dt}\right] \\
\frac{d\Omega'}{dt} &= D + E \cos 2\omega \\
\frac{dM'}{dt} &= F + G \cos 2\omega
\end{aligned}\right\} \quad (9)$$

where

$$A = \frac{3n}{32}\left[\frac{R_e}{p}\right]^4 J_{20}^2 \,[(1-e^2)e(1-16\cos^2 i + 15\cos^4 i)]$$

$$B = -\frac{3n}{128}\left[\frac{R_e}{p}\right]^4 J_{20}^2[10 - 24\sqrt{1-e^2} + 25e^2 + (36+192\sqrt{1-e^2}-126e^2)\cos^2 i-(430 + 360\sqrt{1-e^2} - 45e^2)\cos^4 i]$$

$$C = -\frac{3n}{64}\left[\frac{R_e}{p}\right]^4 J_{20}^2 \,\{[2+e^2 - 11(2 + 3e^2)\cos^2 i](5\cos^2 i - 1)+ 40\,(2+5e^2)\cos^4 i\}$$

$$D = \frac{3n}{32}\left[\frac{R_e}{p}\right]^2 J_{20}^2 \,[(4 + 12\sqrt{1-e^2} - 9e^2)\cos i - (40 + 36\sqrt{1-e^2} - 5e^2)\cos^3 i]$$

$$E = \frac{3n}{32}\left[\frac{R_e}{p}\right]^2 J_{20}^2 \,[e^2(11 + 25\cos^2 i)\cos i]$$

$$F = \frac{3n}{128}\left[\frac{R_e}{p}\right]^2 J_{20}^2\sqrt{1-e^2}[10+6\sqrt{1-e^2} - 25e^2 - (60 + 96\sqrt{1-e^2} - 90e^2)\cos^2 i + (130 + 144\sqrt{1-e^2} - 25e^2)\cos^4 i]$$

$$G = \frac{-3n}{32}\left[\frac{R_e}{p}\right]^2 J_{20}^2\sqrt{(1-e^2)^3}\,(1-16\cos^2 i + 15\cos^4 i)$$

$$p = a(1-e^2)$$

and A, C, E, and G are coefficients of long–periodic terms and B, D, and F are secular terms.

It is extremely important to note in Equations (9) that all the primed orbital elements are a function of the argument of periapsis ω. The primed differential terms of Equations (9) are to be added to Equations (1) in averaged form such that the combined set will include the nonlinear J_{20}^2 effect in the propagation of mean orbital elements of a, e, i, Ω, ω, and M, with the understanding that R^* in Equations (1) does not include J_{20}^2 terms.

Since we use the nonsingular (in e only) set of Equations (7) for integration purposes, we must also transform the set in Equations (9) from e, ω, and M variables to h, k, and λ_N variables also. To that end, we need the following additional identities to carry out the conversions:

$$\left.\begin{aligned}
\sin 2\omega &= 2hk/(h^2 + k^2) \\
\cos 2\omega &= (h^2 - k^2)/(h^2 + k^2)
\end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned}
\frac{dh'}{dt} &= \frac{h}{e}\frac{de'}{dt} + k\frac{d\omega'}{dt} \\
\frac{dk'}{dk} &= \frac{k}{e}\frac{de'}{dt} - h\frac{d\omega'}{dt} \\
\frac{d\lambda_N'}{dk} &= \frac{1}{S_o}\left[\frac{dM'}{dt} + \frac{d\omega'}{dt}\right] + \frac{d\Omega'}{dt}
\end{aligned}\right\} \quad (11)$$

Combining Equations (9), (10), and (11), we have expressed dh'/dt, dk'/dt, and $d\lambda_N'/dt$ in terms of h, k, and λ_N. When these dh'/dk, dk'/dt, and $d\lambda_N'/dt$ are added, respectively, into dh/dt, dk/dt, and $d\lambda_N/dt$ in Equations (7), we have a complete set of differential equations for a satellite trajectory propagation in mean orbital elements that includes the sought–after nonlinear J_{20}^2 effects, with the understanding that R in Equations (7) does not contain J_{20}^2 terms. This complete set of differential equations will be used in the numerical integration program (called LOPJ2S[8]) for long–term orbit predictions in mean orbital elements, with a cost–effectiveness that is orders of magnitude higher than the cost–effectiveness of a numerical integration program in rectangular coordinates such as ASAP. The accuracy of LOPJ2S is accurate enough for most satellite mission preliminary designs, even for the TOPEX/POSEIDON project that require an OD accuracy of abut 15 cm.

We note here that LOPJ2S results are excellent but still are approximate in the sense that LOPJ2S only includes the nonlinear term J_{20}^2. In theory other nonlinear terms such as $J_{20}{*}J_{n0}$, where n = 2, 3, 4, . . ., should also be included in the differential equations. Due to the fact that $J_{20} >>> J_{n0}$ for Earth we conclude that the nonlinear effects of these cross–product terms are too small to be considered, in addition to the reality that it is extremely difficult to have these cross–terms derived.

III. Sample Accuracy Verification Via Timing Comparisons

Before we make accuracy checks of long–term predictions of the LOPJ2S program in mean elements with respect to the high–precision ASAP program in osculating elements (via cartesian coordinates), we must first deal with the important and practical fact of converting the initial mean elements into the corresponding initial osculating elements, and vice versa. The detailed mathematical formulation of the conversion technique is documented in Reference [9] and will be the subject of a future paper. It is sufficient to mention here that the conversion has a second–order accuracy in the sense that conversion is carried out with the effects of geopotential harmonics of J_{20}, J_{30}, J_{40}, and the nonlinear J_{20}^2 which has about the same order of magnitude as J_{40}. In general, the second–order conversion is accurate enough to assure that the two sets of initial conditions are equivalent such that long–term orbit prediction comparisons between LOPJ2S and ASAP program become meaningful. If a first–order (with J_{20} effect only) conversion method is used the same comparisons usually give results that are erroneous and very difficult to interpret. We have demonstrated in Reference [9] that the second–order conversion technique improves at least an order of magnitude in conversion accuracy over the first–order conversion technique.

Since we claim that the J_{20}^2 effects in the LOPJ2S program contribute significantly in long–term orbit prediction accuracies, the goal of our comparisons is to show that the agreement of corresponding results between the LOPJ2S program in mean elements and the high–precision ASAP program in osculating elements is at least an order of magnitude better than that between the LOP program (with no J_{20}^2 effects) in mean elements and the ASAP program.

We propose to use the "timing deviation" at the nodal point (argument of latitude $u = \omega + f = 0$) of a prescribed revolution (say, at 100th or 10,000th revolution) as the criterion to determine the relative overall orbit prediction accuracy of the LOPJ2S (or LOP) program with respect to the ASAP program. For example if the timing deviation is only a few seconds at the nodal point of 1000th Rev (each nodal period is 2 hours, say) and the nodal deviation is only a few hundredths of a degree, we shall consider the comparison is almost perfect in such long–term orbit prediction in mean elements.

The results of two sample calculations will be shown. The first one is for a frozen orbit in the vicinity of the critical inclination angle since such an orbit is very similar to the future nominal TOPEX/POSEIDON orbit to be selected within the next few months. The second one is for a nonfrozen orbit at a relatively low inclination angle in order to show the applicability of the LOPJ2S program in a variety of orbit selection processes. Each sample computation (using zonals from J_{20} through J_{130}) will compare three long–term trajectory prediction results from (A) LOPJ2S program with J_{20}^2, (B) high precision ASAP program, and (C) LOP program without J_{20}^2.

(1) FROZEN ORBIT SAMPLE COMPARISONS:

(1A) LOPJ2S run in mean elements

Strictly speaking, the concept of the so–called frozen orbit is valid only in terms of "fictitious" mean orbital elements in the sense that the average semi–major axis, eccentricity, and argument–of–periapsis of a satellite orbiting an aspherical planet remain constant during a long–periodic cycle, since the osculating orbital elements of such a satellite always vary instantaneously from point to point in a short–periodic manner. For initial mean elements of $a_o' = 7713.14$ km and $i_o' = 64.8°$ an Earth satellite under the disturbing harmonics of J_{20}, through J_{130}, and J_{20}^2, an argument–of– periapsis ω_o' $= 270.0°$ and an eccentricity of $e_o' =$ 0.00073506 are required, in order to maintain a true frozen orbit. We arbitrarily choose Node $\Omega' = 0.0°$ and true anomaly f' $= 90.0°$. Here the primed quantities denote mean elements. The long–periodic cycle of the frozen orbit is about 19,200 revolutions or 1498.57 days with constant e' and ω', as shown in Table 1A from the results of orbit predictions of the LOPJ2S program that includes the J_{20}^2 effect.

TABLE 1A. FROZEN ORBIT RESULTS FROM LOPJ2S (MEAN ELEMENTS)

(J_{20} THROUGH J_{130}, WITH J^2_{20})

WO = 270.0000 FO = 90.0000 AO = 7713.1400 EO = 0.0007351 IO = 64.80000 NODE = 0.000000

REV.NO	TIME,SEC	NODEL PERIOD	DEL N.P.	ECC,E	A.P.,W	T.A.	LONG.ASC.NODE	S(REV)-S1	NODE	DEL NODE
1	6743.578578	6743.578578	0.0000000	0.0007351	270.00000	90.00000	3.031429	0.0000000	359.82991	-0.1700896
400	2697431.431110	6743.578578	0.0000000	0.0007351	270.00002	89.99998	213.272126	-0.0000001	291.96415	-0.1700896
800	5394862.862219	6743.578578	0.0000000	0.0007351	270.00004	89.99996	35.167562	0.0000002	223.92830	-0.1700896
1200	8092294.293329	6743.578578	0.0000000	0.0007351	270.00006	89.99994	217.062998	0.0000001	155.89244	-0.1700896
⋮										⋮
18000	121384414.399940	6743.578578	0.0000000	0.0007351	269.99994	90.00006	296.671315	-0.0000001	-181.61335	-0.1700896
18400	124081845.831051	6743.578578	0.0000000	0.0007351	269.99996	90.00004	118.566751	0.0000003	-249.64920	-0.1700896
18800	126779277.262161	6743.578578	0.0000000	0.0007351	269.99998	90.00002	300.462188	0.0000000	-317.68505	-0.1700896
19200	129476708.693270	6743.578578	0.0000000	0.0007351	270.00000	90.00000	122.357624	-0.0000001	-25.72090	-0.1700896

(1B) ASAP run in osculating elements

Since ASAP requires osculating orbit elements as input, it is necessary to accurately convert the mean orbital elements of case (1A) into osculating elements. This conversion aided by the recent accurate "second–order" conversion technique presented in Reference [9] is shown below: (1st–order elements listed also for comparisons)

	a_o' or a_o km	e_o' or e_o	i_o' or i_o	Ω_o' or Ω_o	ω_o' or ω_o	f_o' or f_o
mean	7713.14000	0.00073506	64.80000°	0.00000°	270.0000°	90.00000°
osc (2nd)	7720.15906	0.00107267	64.81225°	–.00107°	298.0415°	61.95852°
osc (1st)	7720.15288	0.00079586	64.81225°	–.00007°	309.3124°	50.68705°

The primed quantities denote the mean orbital elements. Note that for mean orbital elements the choice of initial true anomaly f'_0 (or mean anomaly M'_0) is entirely arbitrary since by definition mean elements are obtained by averaging with respect to the mean anomaly. Table 1B shows the osculating elements propagation results from the high–precision ASAP program.

TABLE 1B. FROZEN ORBIT RESULTS FROM ASAP (OSC. ELEMENTS)

(J_{20} THROUGH J_{130}, INITIAL MEAN ELEMENTS $a' = 7713.14$ km, $e' = 0.00073506$, $i'_0 = 64.8°$, $node'_0 = 0.0°$, $w'_0 = 270.0°$, $f'_0 = 90.0°$)

OSC INPUT:A,E,I,N,W,F = 7720.159060 .00107267 64.812250 359.999830 298.041500 61.958520, ASC. NODE PRINT BELOW

REV	T,SEC	A	E	I	NODE	W	W+F	LONG	DELTA S	N.P.	DELTA NP
0	0.0000	7720.159060	.00107267	64.81225	359.99983	298.04150	.200d-04	31.37653			
1	6743.5804	7720.159059	.00107264	64.81225	359.82974	298.03831	.655d-07	3.03125			
400	2697432.2979	7720.158599	.00105868	64.81225	291.96392	296.80143	.102d-05	213.26828	-.197d-06	6743.5806	-.305d-03
800	5394864.6070	7720.158145	.00104223	64.81225	223.92801	295.63528	.704d-07	35.15999	.174d-06	6743.5808	-.305d-04
1200	8092296.9248	7720.157708	.00102350	64.81225	155.89210	294.55864	.957d-06	217.05166	-.389d-06	6743.5809	.610d-04
⋮											⋮
7200	48553782.1487	7720.156595	.00069523	64.81225	215.35493	301.27091	.477d-06	65.42634	-.789d-07	6743.5808	.305d-04
7600	51251214.4882	7720.156946	.00069136	64.81226	147.31923	303.47831	.659d-06	247.31813	.735d-06	6743.5807	-.610d-04
8000	53948646.8202	7720.157333	.00069207	64.81226	79.28354	305.71279	.419d-06	69.20996	.698d-06	6743.5808	.000d+00
8400	56646079.1485	7720.157750	.00069733	64.81226	11.24785	307.89534	.620d-06	251.10182	.860d-06	6743.5808	.000d+00
⋮											⋮
18000	121384453.5642	7720.160324	.00109757	64.81225	178.38947	301.76666	.615d-05	296.51050	.238d-05	6743.5807	-.610d-04
18400	124081885.8468	7720.159882	.00109110	64.81225	110.35356	300.39545	.125d-05	118.40232	-.103d-05	6743.5808	.305d-04
18800	126779318.1377	7720.159427	.00108185	64.81225	42.31765	299.05924	.401d-05	300.29411	.496d-06	6743.5806	-.183d-03
19200	129476750.4325	7720.158965	.00106990	64.81225	334.28174	297.77137	.541d-05	122.18587	-.758d-06	6743.5806	-.183d-03

(1C) LOP run in mean elements

In order to see the J_{20}^2 effect more clearly we also input the same initial mean elements from Table 1A into the LOP program that does not consider the J_{20}^2 effect. The results are shown in Table 1C.

TABLE 1C. FROZEN ORBIT RESULTS FROM LOP (MEAN ELEMENTS)

(J_{20} THROUGH J_{130}, NO J^2_{20})

WO = 270.0000 FO = 90.0000 AO = 7713.1400 EO = 0.0007351 IO = 64.80000 NODE = 0.000000

REV.NO	TIME,SEC	NODEL PERIOD	DEL N.P.	ECC,E	A.P.,W	T.A.	LONG.ASC.NODE	S(REV)-S1	NODE	DEL NODE
1	6743.579042	6743.579042	0.0000000	0.0007351	269.99998	90.00002	3.031409	0.0000000	359.82989	-0.1701071
400	2697431.616688	6743.579042	0.0000002	0.0007351	269.99216	90.00784	213.264344	-0.0000001	291.95714	-0.1701071
800	5394863.233404	6743.579042	0.0000002	0.0007350	269.98446	90.01554	35.151998	0.0000002	223.91428	-0.1701071
1200	8092294.850147	6743.579042	0.0000003	0.0007350	269.97702	90.02298	217.039652	0.0000001	155.87142	-0.1701071
⋮										⋮
18000	121384422.750649	6743.579042	-0.0000001	0.0007350	270.02154	89.97846	296.321145	0.0000000	-181.92863	-0.1701071
18400	124081854.367258	6743.579042	0.0000000	0.0007350	270.01404	89.98596	118.208799	0.0000002	-249.97149	-0.1701071
18800	126779285.983895	6743.579042	0.0000000	0.0007351	270.00629	89.99371	300.096453	-0.0000001	-318.01434	-0.1701071
19200	129476717.600560	6743.579042	0.0000001	0.0007351	269.99845	90.00155	121.984107	0.0000000	-26.05720	-0.1701071

In Tables 1A through 1C the nodal revolution number is the independent variable. To avoid ambiguities, the revolution number is defined here. We always start with the initial zeroth Rev which always contains the initial conditions (whether starts at the node or not) and ends with the first nodal crossing. Accordingly, for the first Rev the time in these tables is the start (or the end of the zeroth Rev) of first Rev. The frozen orbit in Table 1A for the LOPJ2S run has constant mean eccentricity e and argument–of– periapsis ω with a long–periodic cycle of about 19,200 revs or 129, 476,708.69 sec. (1498.5730 days). The nodal period remains absolutely constant during the whole long–periodic cycle at 6743.578578^s. At the end of the long–periodic cycle (i.e., at the start of the 19,200th REV), the satellite is at the ascending node point at $\Omega = 334.2791°$. The column header with "S(REV)–S$_1$" denotes the deviation between the current fundamental interval and the first fundamental interval, thus, we observe the fundamental interval also remains constant throughout the long–periodic cycle. The last column "DELNODE" denotes the deviation of the current node from the preceding node, and thus the node regression per revolution remains constant also throughout the long–periodic cycle.

In Table 1B for the ASAP run, we note a and i columns are, respectively, almost constant and the column $\omega + f = u$ is nearly 360.0 or zero, because each and every line is at a nodal point. In addition, the eccentricity e goes through a periodic cycle from the initial value to the minimum and then rises back to the initial value. At 19,200 Rev the time is 129,476,750.43^s (1498.5735 days). Nodal periods are almost constant at 6743.5806^s with an average deviation of about 0.0005^s.

Table 1C has the same format as Table 1A and thus needs no further explanations. We note, however, in Table 1C that the mean argument–of– periapsis, eccentricity, and nodal period are not as constant as those in Table 1A, because the initial mean eccentricity in Table 1C is obtained from that of the true frozen orbit with J_{20}^2 effects in Table 1A.

The computing speed of either LOP runs or LOPJ2S runs is at least two orders of magnitude faster than that of the corresponding ASAP run.

The table below should summarize the salient points from Tables 1A, 1B, and 1C for purposes of convenient comparisons:

THE FROZEN ORBIT CASE

	REV. NO.	TIME sec	PERIOD sec	NODE deg	LONG. ASC. NODE deg.
LOPJ2S	19,200	129,476,708.693 (1498.5730 days)	6743.57857	334.2791	122.4576
ASAP	19,200	129,476,750.433 (1498.5735 days)	6743.58060 6743.58074, avg	334.2817	122.1859
LOP	19,200	129,476,717,600 (1498.5731 days)	6743.57904	333.9428	121.9841

At the same nodal point at the start of 19,200th Rev, we obtain from the above table the following deviations of LOPJ2S and LOP results with respect to the precision ASAP result:

THE FROZEN ORBIT CASE

	TIME, sec (at 19,200th Rev)	PERIOD, sec	NODE, deg	LONG ASC. NODE, deg
LOPJ2S deviations	−41.74	−0.00217 (avg)	−0.0026 (−0.29 km)	0.1717 (19.11 km)
LOP deviations	−32.83	−0.00170 (avg)	−0.3389 (−37.73 km)	−0.2018 (−22.46 km)

From the above table we observe in this sample case for a frozen orbit that the most significant improvement of LOPJ2S over LOP is in the node parameter due to the J_{20}^2 effect, since the node deviation between LOPJ2S and ASAP is almost vanishing (−0.0026°), but that between LOP and ASAP is −0.34°. Thus, it shows that the improvement of LOPJ2S results over LOP results is at least two orders in magnitude. This is important in the inertial frame that the cross–track deviation is almost eliminated due to the addition of the J_{20}^2 effect in LOPJ2S to obtain correct rates of nodal regression or progression. In the rotating meridian frame, the deviation of the longitude of ascending node between LOPJ2S and ASAP and that between LOP and ASAP are about the same in magnitude but opposite in direction. It thus appears that in this case LOPJ2S might not have offered anyu improvement. To resolve this dilemma we look into timing deviations for help. Note that timing here is directly related to nodal periods which are in turn closely related to the accuracy of initial orbital elements conversions[9]. Any slight inaccuracy in an initial conversion will be reflected in nodal periods and thus later in timing. In order to match the longitude of node from LOPJ2S to that of ASAP, the LOPJ2S timing must be increased by 41.74^s in 19,200 revs. In 41.74^s the Earth will rotate exactly 0.1717° (19.11 km) and node will shift only about 0.001°. This means that LOPJ2S, in order to correct the timing error of 41.74^s, would require a nodal period incre se of 0.00217^s which might be achievable by including harmonics higher than J_{40} in an orbital elements conversion method, i.e., by a third–order conversion method which might not be accomplished in practical sense due to the extreme mathematical complexities involved and to associated unrealistic lengthy computations. This whole illustration here is to show (a) our clear understanding of the problem, and (b) that the accuracy we have achieved in reality now by the second–order conversion method[9] is sufficiently accurate for most satellite mission design purposes. The reason is that we are using here a sample illustration for a frozen orbit with a long–periodic cycle of 19,200 revs (1498.57 days). In practical cases, only one hundred days or so are needed before a maneuver is necessary to correct orbit parameter deviations due to other small perturbations resulting from such sources as the non–gravitational force of atmospheric drag. For one hundred days or so the cumulative error due to the use of LOPJ2S without drag is, in general, negligible. In fact, we repeat that the extremely small deviations between LOPJ2S and ASAP here after a long duration of trajectory propagation are entirely due to slight errors in the initial orbital elements

TABLE 1D. FROZEN ORBIT RESULTS FROM LOPJ2S (MEAN ELEMENTS)

(J_{20} THROUGH J_{130} WITH J^2_{20}), (a INCREASED BY 1.56 m TO MATCH RESULTS OF THE ASAP IN TABLE 1B)

WO = 270.0000 FO = 90.0000 AO = 7713.1416 EO = .0007351 IO = 64.80000 NODE = 0.000000

REV.NO	TIME,SEC	NODEL PERIOD	DEL N.P.	ECC,E	A.P.,W	T.A.	LONG.ASC.NODE	S(REV)-S1	NODE	DEL NODE
1	6743.580741	6743.580741	0.0000000	.0007351	270.00000	90.00000	3.031420	0.0000000	359.82991	-.1700896
400	2697432.296312	6743.580741	.0000002	.0007351	269.99960	90.00040	213.268540	-.0000001	291.96418	-.1700896
800	5394864.592625	6743.580741	.0000002	.0007351	269.99921	90.00079	35.160391	.0000002	223.[illegible]2835	-.1700896
1200	8092296.888940	6743.580741	.0000002	.0007351	269.99883	90.00117	217.052241	.0000000	155.89253	-.1700896
⋮										⋮
18000	121384453.334027	6743.580741	.0000002	.0007351	270.00109	89.99891	296.509955	.0000002	-181.61204	-.1700896
18400	124081885.630335	6743.580741	.0000001	.0007351	270.00071	89.99929	118.401806	-.0000001	-249.64786	-.1700896
18800	126779317.926645	6743.580741	.0000002	.0007351	270.00031	89.99969	300.293656	.0000000	-317.68368	-.1700896
19200	129476750.222956	6743.580741	.0000002	.0007351	269.99991	90.00009	122.185506	.0000003	-25.71951	-.1700896

conversion, and there appear to be no detectable errors in the mathematical formulation of including the J_{20}^2 effect in LOPJ2S. It will become even more evident in what to be shown in the next paragraph.

In an effort to demonstrate the situation, we shall increase only the semi–major axis a of Table 1A for LOPJ2S 1.56 meters from 7713.14 to 7713.14156 km in order to increase the period slightly (0.00217^s) such that the LOPJ2S timing will catch up with the timing of ASAP. The results come out as expected and are shown in Table D.

The comparison between results of Table 1D from the LOPJ2S run and those of Table 1B from the corresponding ASAP run at 19,200 Rev are tabulated below:

	REV. NO.	TIME sec	PERIOD sec	NODE deg	LONG. ASC. NODE, deg.
LOPJ2S	19,200	129,476,750.223	6743.5807	334.2805	122.1855
ASAP	19,200	129,476,750.433	6743.5807	334.2817 (Avg)	122.1855

The perfect matches between LOPJ2S (with adjusted semi–major axis) and ASAP trajectories for every parameter in the above table are truly remarkable, especially for a duration of one long–periodic cycle of about 1500 days. In general, for frozen orbits the agreements between LOPJ2S and ASAP are accurate enough even without slightly adjusting semi–major axis a if the duration of trajectory propagation is a small fraction of a long–periodic cycle.

Figures 1a and 1b show, respectively, the delta node and delta longitude of ascending node between Table 1B for ASAP and Table 1D for LOPJ2S for a complete long–periodic cycle.

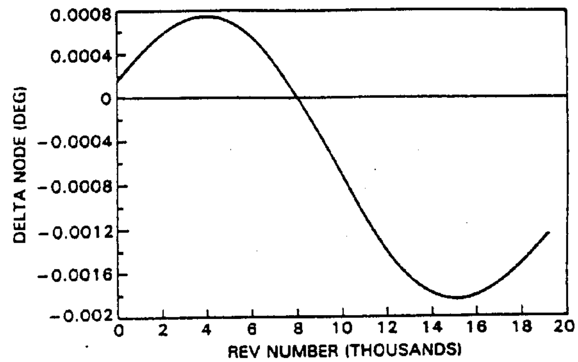

Fig. 1a Delta node between LOPJ2S and ASAP.

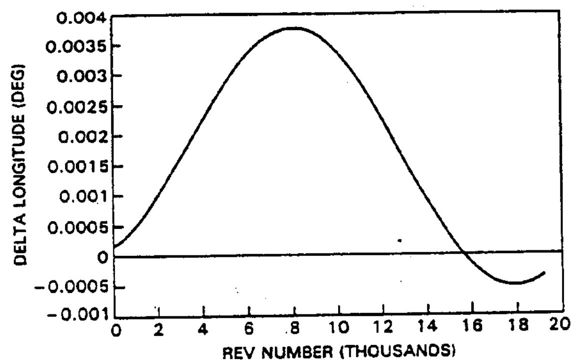

Fig. 1b Delta longitude between LOPJ2S and ASAP.

(2) NONFROZEN ORBIT SAMPLE COMPARISONS:

(2A) A nonfrozen orbit with initial mean orbital elements a' = 7711.92 km, e' = 0.00154025, i' = 24.0°, and ω' = 0.0° is run by the LOPJ2S program. The propagation results of the long–periodic cycle of about 44 days are shown in Table 2A which indicates that e' goes through a cyclic change and ω' goes through a 360° variation in one long–periodic cycle. Note at the 500th Rev in Table 2A the time is about 38.84 days Also note that due to nonfrozen nature of the orbit, the nodal periods in Table 2A is no longer constant but varies in a long–periodic manner with a one half of a long–periodic cycle at about the 283rd Rev (almost 22 days).

TABLE 2A. NONFROZEN ORBIT RESULTS FROM LOPJ2S (MEAN ELEMENTS)

(J_{20} THROUGH J_{130}, WITH J^2_{20})

WO = 90.0000 FO = 180.0000 AO = 7711.9200 EO = 0.0015402 IO = 24.00000 NODE = 0.000000

REV.NO	TIME,SEC	NODEL PERIOD	DEL N.P.	ECC,E	A.P.,W	T.A.	LONG.ASC.NODE	S(REV)-S1	NODE	DEL NODE
1	1683.895827			0.0015402	90.11724	269.88276	24.249574		359.90833	
50	331081.142499	6722.386131	-0.0136491	0.0014961	113.19850	246.80150	70.070527	-0.0000580	341.97542	-0.3659771
100	667200.166397	6722.376027	-0.0237529	0.0013669	137.72956	222.27044	87.441471	-0.0001013	323.67659	-0.3659760
150	1003318.859776	6722.373113	-0.0266667	0.0011722	164.68068	195.31932	104.813849	-0.0001144	305.37782	-0.3659751
200	1339437.614768	6722.378270	-0.0215096	0.0009503	196.59307	163.40693	122.186005	-0.0000934	287.07908	-0.3659746
250	1675556.804887	6722.389940	-0.0098401	0.0007725	237.23879	122.76121	139.556351	-0.0000445	268.78034	-0.3659748
300	2011676.672359	6722.404606	0.0048260	0.0007393	285.69139	74.30861	156.923844	0.0000175	250.48159	-0.3659755
350	2347797.255788	6722.417869	0.0180895	0.0008760	329.92087	30.07913	174.288300	0.0000740	232.18279	-0.3659766
400	2683918.379540	6722.425770	0.0259905	0.0010924	4.55327	355.44673	191.650443	0.0001081	213.88393	-0.3659777
450	3020039.706968	6722.425959	0.0261790	0.0013027	32.92074	327.07926	209.011686	0.0001097	195.58503	-0.3659785
500	3356160.840884	6722.418379	0.0185989	0.0014586	58.11893	301.88107	226.373711	0.0000783	177.28610	-0.3659787

(2B) To obtain the corresponding osculating elements propagation from the ASAP program, the 2nd–order conversion from the initial mean elements from Table 2A to a corresponding set of initial osculating elements according to Reference [9] is tabulated below: (1st–order elements are listed also for comparisons)

	a_0' or a_0 km	e_0' or e_0	i_0' or i_0	Ω_0' or Ω_0	ω_0' or ω_0	f_0' or f_0
mean	7711.92000	0.00154025	24.00000°	0.0°	90.0°	180.0°
osc (2nd)	7710.49842	0.00085610	23.98824°	0.0°	90.0°	180.0°
osc (1st)	7710.49165	0.00085778	23.98813°	0.0°	90.0°	180.0°

The osculating elements propagation from the ASAP program is shown in Table 2B.

TABLE 2B. NONFROZEN ORBIT RESULTS FROM ASAP (OSC. ELEMENTS)

(J_{20} THROUGH J_{130}, INITIAL MEAN ELEMENTS: $a_0' = 7711.92$ km, $e_0' = 0.00154025$, $i_0' = 24.0^\circ$, $node_0' = 0.0^\circ$, $w_0' = 90.0^\circ$, $f_0' = 180.0^\circ$)

OSC INPUT:A,E,I,N,W,F = 7710.498420 0.00085610 23.988240 0.000000 90.000000 180.000000, ASC. NODE PRINT BELOW

REV	T,SEC	A	E	I	NODE	W	W+F	LONG	DELTA S	N.P.	DELTA NP
1	1683.9028	7713.347891	0.00183040	24.01185	359.90874	57.34738	0.323E-10	24.24996			
50	331081.0347	7713.337818	0.00143349	24.01185	341.97550	73.74274	0.121E-05	70.07105	0.444E-04	6722.3836	-0.138E-01
100	667199.9426	7713.330566	0.00092114	24.01186	323.67627	91.30077	0.770E-05	87.44209	0.100E-03	6722.3740	-0.235E-01
150	1003318.5180	7713.328516	0.00034170	24.01189	305.37706	114.17635	0.106E-05	104.81452	0.988E-04	6722.3706	-0.269E-01
200	1339437.1555	7713.332278	0.00028040	24.01193	287.07789	286.55212	0.411E-05	122.18674	0.898E-04	6722.3759	-0.215E-01
250	1675556.2279	7713.340732	0.00086378	24.01195	268.77877	311.52863	0.234E-04	139.55721	0.388E-04	6722.3876	-0.983E-02
300	2011675.9786	7713.351353	0.00138568	24.01197	250.47969	329.24274	0.125E-04	156.92485	-0.129E-04	6722.4023	0.491E-02
350	2347796.4465	7713.360959	0.00180247	24.01196	232.18062	345.99642	0.119E-05	174.28952	-0.806E-04	6722.4157	0.183E-01
400	2683917.4564	7713.366665	0.00208173	24.01194	213.88155	2.45321	0.841E-06	191.65192	-0.111E-03	6722.4234	0.259E-01
450	3020038.6690	7713.366752	0.00220187	24.01191	195.58246	18.80470	0.122E-05	209.01345	-0.111E-03	6722.4236	0.262E-01
500	3356159.6882	7713.361193	0.00215374	24.01187	177.28331	35.14475	0.155E-05	226.37575	-0.904E-04	6722.4160	0.186E-01

(2C) To see the effect of missing J_{20}^2 terms in LOP, we use the same initial mean elements from Table 2A and obtain the LOP propagation results shown in Table 2C.

TABLE 2C. NONFROZEN ORBIT RESULTS FROM LOP (MEAN ELEMENTS)

(J_{20} THROUGH J_{130}, NO J^2_{20})

WO = 90.0000 FO = 180.0000 A = 7711.9200 EO = 0.0015402 IO = 24.00000 NODE = 0.000000

REV.NO	TIME,SEC	NODEL PERIOD	DEL N.P.	ECC,E	A.P.,W	T.A.	LONG.ASC.NODE	S(REV)-S1	NODE	DEL NODE
1	1683.905770			0.0015402	90.11680	269.88320	24.249733		359.90853	
50	331083.099644	6722.425909	-0.0135655	0.0014962	113.11065	246.88935	70.101634	-0.0000576	342.01471	-0.3651795
100	667204.113777	6722.415846	-0.0236291	0.0013675	137.54384	222.45616	87.504145	-0.0001008	323.75576	-0.3651784
150	1003324.797531	6722.412897	-0.0265781	0.0011731	164.37444	195.62556	104.908089	-0.0001140	305.49687	-0.3651774
200	1339445.539268	6722.417949	-0.0215257	0.0009512	196.12102	163.87898	122.311826	-0.0000934	287.23801	-0.3651770
250	1675566.709918	6722.429484	-0.0099905	0.0007721	236.56017	123.43983	139.713780	-0.0000451	268.97916	-0.3651771
300	2011688.551681	6722.444045	0.0045700	0.0007361	284.93333	75.06667	157.112907	0.0000165	250.72029	-0.3651778
350	2347811.106059	6722.457286	0.0178111	0.0008706	329.26307	30.73693	174.509011	0.0000729	232.46137	-0.3651789
400	2683934.202427	6722.465275	0.0258003	0.0010865	3.96505	356.03495	191.902797	0.0001073	214.20240	-0.3651800
450	3020057.509491	6722.465648	0.0261736	0.0012975	32.33547	327.66453	209.295653	0.0001097	195.94337	-0.3651808
500	3356180.633659	6722.458295	0.0188206	0.0014551	57.49548	302.50452	226.689246	0.0000792	177.68433	-0.3651811

The following Table summarizes the results of Tables 2A, 2B, and 2C in order to facilitate comparisons for various parameters.

NONFROZEN ORBIT CASE

	REV. NO.	TIME sec	NODE deg	LONG. ASC. NODE, deg.	PERIOD (not constant)
LOPJ2S	500	3,356,160.84 (38.84445 days)	177.2861	226.3737	6722.4138 6722.4269 (max at 430 Rev) 6722.3730 (min at 140 Rev)
ASAP	500	3,356,159.69 (38.84444 days)	177.2833	226.3757	6722.4160 6722.4247 (max at 430 Rev) 6722.3706 (min at 140 Rev)
LOP	500	3,356,180.63 (38.84468 days)	177.6843	226.6892	6722.4583 6722.4665 (max at 430 Rev) 6722.4128 (min at 140 Rev)

At the same nodal point at the start of the 500th Rev, the following deviations of LOPJ2S and LOP results, with respect to the precision ASAP results, are obtained from the above table:

NONFROZEN ORBIT CASE

	TIME, sec (at 500th Rev)	PERIOD, sec	NODE, deg	LONG. Asc. Node, deg
LOPJ2S deviations	1.15	–0.0022	0.0026 (0.31 km)	–0.0021 (–0.23 km)
LOP deviations	20.94	–0.0423	–0.4010 (44.64 km)	0.3135 (34.9 km)

The preceding table shows that for the nonfrozen orbit case (for only 39 days) the improvements of the parameter results of the LOPJ2S program run are overwhelmingly better than those of the corresponding LOP program run. Of course, to correct the time error of 1.15^s, the mean semi–major axis a of LOPJ2S has to be reduced slightly in order to match the longitude of ascending node values from the LOPJ2S and ASAP runs, as we have shown in the similar frozen orbit case earlier. Thus it remains true that the orbit prediction accuracy of LOPJ2S is about two orders of magnitude better than that of LOP and the computing speed of LOPJ2S is about two orders of magnitude faster than that of ASAP.

V. DISCUSSIONS AND CONCLUSIONS

We have demonstrated in Reference [9] that the second–order conversion technique using the harmonics J_{20}, J_{30}, J_{40}, and J_{20}^2 improves at least an order of magnitude in orbital–elements conversion–accuracy over the first–order method using the harmonic J_{20} alone. It becomes clear that we could further improve the conversion accuracy even more by including "third–order" harmonics J_{50}, J_{60}, etc., in the conversion technique. It thus implies that the "second–order" conversion technique still introduces some initial orbit–elements conversion errors, albeit extremely small, which could partly affect the future long–term orbit prediction accuracies.

The other part that could also affect the prediction accuracies can be explained mathematically in terms of the effect of cross–harmonic product terms $J_{20}*J_{n0}$, $n \geq 2$. In our expansion method or averaging method, we have taken care of the nonlinear term J_{20}^2 but neglected cross–product terms such as $J_{20}*J_{30}$, $J_{20}*J_{40}$, etc., However, we remark that these cross–product effects on orbit prediction accuracies are negligibly small by comparisons with the effect mentioned in the preceding.paragraph.

We state here that it is not meaningful to compare accuracies of long–term orbit predictions of a LOPJ2S program run with respect to the corresponding high–precision ASAP program run by comparing at any prescribed instant the equivalence of respective set of orbital elements from the two programs, after a long duration of orbit propagations, via the orbital–elements conversion method of Reference [9]. The reason is that any slight deviation in timing or position between the two programs after a long duration of trajectory propagations would upset the equivalence of the orbital–elements conversion results and thus be subject to ambiguously difficult interpretations and comparisons. Accordingly, our concensus is that the timing comparison technique at a designated point (e.g., node) in a prescribed revolution near the end of a long–periodic cycle, illustrated in the preceding section, is the most meaningful way to gauge the long–term orbit prediction accuracy of a LOPJ2S program run with respect to the corresponding high–precision ASAP program run. In addition, any slight time deviation can be easily corrected, if necessary, by slightly adjusting the initial nodal period (via adjusting the semi–major axis a) of either LOPJ2S or ASAP program run through an iterative cut–and–try method or some other quick and simple graphic or analytic partial method.

In conclusion, we successfully claim that the incorporation of the J_{20}^2 term effect in the LOPJ2S program has accomplished the goal that long–term orbit predictions using LOPJ2S formulation in mean elements can be achieved with accuracies almost as close as those using the high–precision ASAP formulation in osculating elements. The LOPJ2S formulation can also be used in a way to supply the conventionally sought–after information on cross–track, along–track, and radial component errors. The orbit prediction accuracy of LOPJ2S is about two orders of magnitude better than that of LOP and the computing speed of the LOPJ2S program is at least two orders of magnitude faster than that of the high–precision ASAP program. For any short–term orbit prediction, the ASAP program must be used.

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